Decomposition Theorem and Göttsche's Formula

§1 Stratifications and semi-small maps

Conventions:
- All varieties are over $\mathbb{C}$
- λ partition of n, $\lambda \vdash n$, write $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)})$
  or $\lambda = (1^{\alpha_1}, 2^{\alpha_2}, \dots, n^{\alpha_n})$ $\quad (\sum \alpha_i = \ell(\lambda) = \ell)$
- $S_\lambda := S_{\alpha_1} \times \dots \times S_{\alpha_n}$
- $X^\lambda := X^{\ell(\lambda)}$, $\quad X^{(\lambda)} = X^\lambda / S_\lambda \cong X^{(\alpha_1)} \times \dots \times X^{(\alpha_n)}$.
- $\rho_n : X^{[n]} \to X^{(n)}$ Hilbert-Chow-morphism.

Def: $\lambda \vdash n$, $\Delta_n \subseteq X^n$ (small) diag. and $q : X^n \twoheadrightarrow X^{(n)}$ quotient. Define

$$X_{\bar{\lambda}}^{(n)} := q(\Delta_{\lambda_1} \times \dots \times \Delta_{\lambda_\ell}) \qquad X_\lambda^{(n)} := X_{\bar{\lambda}}^{(n)} \setminus \Big(\bigcup_{\lambda < \mu} X_{\bar{\mu}}^{(n)} \Big)$$

Properties:
- As points $X_\lambda^{(n)} = \{ \sum \lambda_i x_i \mid x_i \neq x_j,\ i \neq j \}$.
- $X^{(n)} = \coprod_{\lambda \vdash n} X_\lambda^{(n)}$ is a stratification.
- $X^\lambda \longrightarrow X^{(n)} : (x_1, \dots, x_\ell) \longmapsto \sum \lambda_i \cdot x_i$ factors through

$$\begin{tikzcd} X^\lambda \arrow[r] \arrow[d] & X^{(n)} \\ X^{(\lambda)} \arrow[ur, "\lambda"'] & \end{tikzcd}$$

If X sm. q-proj. surface Then λ induces isom $\lambda : X_*^{(\lambda)} := (X^\lambda \setminus \text{big diag}) / S_\lambda$
$\xrightarrow{\sim} X_\lambda^{(n)}$ Also $X_\lambda^{(n)}$ is smooth of dim $2\ell(\lambda)$.

Prop: X sm. q. proj. surface, $\xi \in X_\lambda^{(n)}$, then $\rho_n^{-1}(\xi)$ is irred. of dim $n - \ell(\lambda)$.

Proof: Briançon's theorem (Renjie's talk) $\Rightarrow$ Prop for $\lambda = (n)$ (then $\xi = n \cdot x$ and $\rho_n^{-1}(\xi)$ is the punctual Hilbert scheme irred. of dim $n-1$)

In general, write $\xi = \sum \lambda_i \cdot x_i$, then $\rho_{\lambda_1}^{-1}(\lambda_1 x_1) \times \ldots \times \rho_{\lambda_\ell}^{-1}(\lambda_\ell x_\ell) \overset{\text{"sum"}}{\rightsquigarrow} \rho_n^{-1}(\xi)$

$\Rightarrow \rho_n^{-1}(\xi)$ is irred. of dim $\sum(\lambda_i - 1) = n - \ell(\lambda)$. □

Def: Let $f: X \to Y$ proper morph. of alg. var. f is called <u>semi small</u> (s.s.) if $\exists$ strat. $Y = \coprod Y_t$ st. $\forall t$, $\forall y \in Y_t \cap f(X)$ $2\dim f^{-1}(y) \leq \dim(X) - \dim(Y_t)$

Properties: For surjective f.

- s.smallness can be checked on the strat. $Y = \coprod_{\delta \in \mathbb{Z}_{\geq 0}} Y_f^\delta$ where $Y_f^\delta = \{y \in Y \mid \dim f^{-1}(y) = \delta\}$
- f ss. $\Rightarrow \dim(X) = \dim(Y)$
- X is irred., then f is ss. $\Leftrightarrow \dim X \times_Y X = \dim(X)$.

Def: Let $Y = \coprod Y_t$ be a strat. It is called <u>a stratification for $f: X \to Y$</u> if $\forall t$ $f: f^{-1}(Y_t) \to Y_t$ is a top. fibre bundle.

- Let $Y = \coprod_{t \in T} Y_t$ a strat. for f. Call $A = \{a \in T \mid \forall y \in Y_a,\ 2\dim f^{-1}(y) = \dim(X) - \dim(Y_a)\}$ the set of relevant strata for f.

Prop: X sm. q. proj. surface, then $\rho: X^{[n]} \to X^{(n)}$ is semismall and $\{X_\lambda^{(n)}\}_{\lambda \vdash n}$ is a rel. strat. for ρ.

Proof: By the Prop before, ρ is ss. and we are left to check $\{X_\lambda^{(n)}\}_{\lambda \vdash n}$ is a strat. for ρ. This is in FGA explained. Idea: Show this for $\rho^{-1}(X_{(n)}^{(n)}) \to X_{(n)}^{(n)}$

General case will be a product of loc. triv. maps. $\square$

§2 Intersection Theory

Recall: $\cdot$
$$\begin{array}{ccc} X_1 \times_Y X_2 & \xrightarrow{p_2} & X_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ X_1 & \xrightarrow{f} & Y \end{array}$$
f proper $\rightsquigarrow \Gamma \in A_d(X_1 \times_Y X_2)$ correspondence (some $d \in \mathbb{Z}$)
$\rightsquigarrow \Gamma_* : A_k(X_1) \longrightarrow A_{k+d-\dim(X_1)}(X_2) : \alpha \mapsto (p_2)_*(p_1^*(\alpha) \cap \Gamma)$

- $\Gamma_1 \in A_*(X_1 \times_Y X_2)$, $\Gamma_2 \in A_*(X_2 \times_Y X_3)$ $\quad p_{ij} : X_1 \times_Y X_2 \times_Y X_3 \longrightarrow X_i \times_Y X_j$. Define
  $\Gamma_2 \circ \Gamma_1 := (p_{13})_*(p_{12}^* \Gamma_1 \cap p_{23}^* \Gamma_2) \in A_*(X_1 \times_Y X_3)$ And $(\Gamma_2 \circ \Gamma_1)_* = (\Gamma_2)_* \circ (\Gamma_1)_*$
- ${}^t(-) : A_*(X_1 \times_Y X_2) \longrightarrow A_*(X_2 \times_Y X_1)$

Rmk: $A_*(X \times_Y X)$ is a ring with unit Δ_X and involution ${}^t(-)$.

Fact: These constructions all work on cohomological correspondences $\Gamma \in H^*(X_1 \times_Y X_2, \mathbb{Z})$

Generalize: All of above works also for quotient varieties X/G (X smooth, G finite group) provided $\mathbb{Q}$-coeff. are taken.

§3 Decomposition Theorem

Setup: $f : X \to Y$ proper surj. s.s. X quotient var. Let $\{Y_a\}_{a \in A}$ be a rel. strat. of f, fix $y_a \in Y_a$, $t_a := \frac{1}{2}(\dim(X) - \dim(Y_a))$. $E_a := \{$irred. comps of $f^{-1}(y_a)\}$ is a right $G_a := \pi_1(Y_a, y_a)$ set. Using Stein factorization + Zariski's connectedness Theorem $\rightsquigarrow \underbrace{\coprod_i V_{a,i}}_{= V_a} : \underbrace{\coprod_{i \in G_a\text{-orbits of } E_a} Z_{a,i}}_{= Z_a} \longrightarrow Y_a$ finite morph.

Thm: Assume $\exists$ quotient var. $\overline{Z_{a,i}}$ s.t. $Z_{a,i} \hookrightarrow \overline{Z_{a,i}}$ is dense open. and $\exists$ $\overline{\nu_{a,i}} : \overline{Z_{a,i}} \to \overline{Y_a}$ extending $\nu_{a,i}$ "with same properties". Then $\exists$ correspondence $\overline{\Gamma} = (\overline{\Gamma_a})_{a\in A} \in \bigoplus_{a\in A} A_{t_a + \dim(\overline{Z_a})}(\overline{Z_a} \times_Y X)$ together with an inverse correspondence $\overline{\Gamma}'$ inducing isom $\overline{\Gamma}_* : \bigoplus_{a\in A} A_*(\overline{Z_a}) \xrightarrow{\sim} A_*(X)$ $\left((\overline{\Gamma_a})_* : A_*(\overline{Z_a}) \to A_{*+t_a}(X) \right)$

Cor: $\exists$ correspondence $\Gamma = (\Gamma_\lambda)_{\lambda\vdash n} \in \bigoplus_{\lambda \vdash n} A_{n+\ell(\lambda)}(X^{(\lambda)} \times_{X^{(n)}} X^{[n]})$ (with inverse Γ') inducing isom $\Gamma_* : \bigoplus_{\lambda\vdash n} A_*(X^{(\lambda)}) \longrightarrow A_*(X^{[n]})$ $\quad (\Gamma_\lambda)_* : A_{*+\ell(\lambda)-n}(X^{(\lambda)}) \to A_*(X^{[n]})$.

Proof: Apply the Thm to ρ. $\square$

Cor: (Göttsche's Formula) X smooth q.proj. surface, $b_i(X) := \dim H^i(X, \mathbb{Q})$ and $P_X(t) = \sum_{m\geq 0} b_m(X) t^m$, then

$$\sum_{n\geq 0} q^n \cdot P_{X^{[n]}}(t) \overset{(*)}{=} \prod_{m\geq 1} \frac{(1+t^{2m-1}q^m)^{b_1(X)} (1+t^{2m+1}q^m)^{b_3(X)}}{(1-t^{2m-2}q^m)^{b_0(X)} (1-t^{2m}q^m)^{b_2(X)} (1-t^{2m+2}q^m)^{b_4(X)}}$$

Proof: Apply the ~~Thm~~ cycle class map to the Γ from the Thm. $\rightsquigarrow$ cohom. correspondence inducing isom $\bigoplus_{\lambda\vdash n} H^*(X^{(\lambda)}, \mathbb{Q}) \xrightarrow{\sim} H^*(X^{[n]}, \mathbb{Q})$

Computing indices... $\rightsquigarrow$ $\sum_{\lambda\vdash n} b_{k-2n+2\ell(\lambda)}(X^{(\lambda)}) = b_k(X^{[n]})$

Plug this in to LHS(*)

$$LHS(*) = \sum_{n\geq 0} q^n \sum_{\lambda\vdash n} \sum_{m\geq 0} b_{m-2n+2\ell(\lambda)}(X^{(\lambda)})\, t^{m-2n+2\ell(\lambda)}\, t^{2n-2\ell(\lambda)}$$

$$= \sum_{n\geq 0} q^n \sum_{\lambda \vdash n} P_{X^{(\lambda)}}(t)\, t^{2n-2\ell(\lambda)} = \sum_{n\geq 0} q^n \sum_{\lambda\vdash n} t^{2n-2\ell(\lambda)} P_{X^{(\alpha_1)}}(t) \cdots P_{X^{(\alpha_n)}}(t)$$

$$= \sum_{n\geq 0} \sum_{\lambda\vdash n} P_{X^{(\alpha_1)}}(t)\, q^{\alpha_1} P_{X^{(\alpha_2)}}(t) (q^2 t^2)^{\alpha_2} \cdots P_{X^{(\alpha_n)}}(t) \left(q^n t^{2(n-1)}\right)^{\alpha_n}$$

$$= \prod_{m\geq 1} \sum_{n\geq 0} P_{X^{(n)}}(t) \left(q^m t^{2(m-1)}\right)^n = \text{RHS}(*)$$ Pablo's talk (MacDonald Formula)

$\square$