∞-CATEGORIES SEMINAR
TALK III. ∞-CATEGORIES

I. Morphisms of simplicial sets

Recall that last week we defined a simplicial set as follows.

Definition 1. A *simplicial set* is a sequence of sets $K_0, K_1, K_2, \dots$ together with maps $d_i : K_q \to K_{q-1}$ and $s_i : K_q \to K_{q+1}$ for $0 \leq i \leq q$ satisfying the axioms:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{if } i < j \\ s_i s_j &= s_{j+1} s_i && \text{if } i \leq j \\ d_i s_j &= s_{j-1} d_i && \text{if } i < j \\ d_j s_j &= \mathrm{id} = d_{j+1} s_j && \\ d_i s_j &= s_j d_{i-1} && \text{if } i > j+1. \end{aligned}$$

The d_i are called *face morphisms*, the s_i are called *degeneracy morphisms*, and the elements of K_q are called q-simplices.

We saw that, heuristically, we can think of the set K_0 as the points of some topological space X, we can think of the set K_1 as the paths, K_2 as morphisms to X from a triangle, K_3 as morphisms to X from a tetrahedron, etc. Similarly, we can think of the face morphisms $d_i : K_3 \to K_2$ (resp. $K_2 \to K_1$, resp. $K_1 \to K_0$) as telling us how to restrict a morphism from a tetrahedron to one of its faces, (resp. a morphism from a triangle to one of its edges, resp. a path to its endpoints), and the maps $s_i : K_0 \to K_1$ (resp. $K_1 \to K_2$, resp. $K_2 \to K_3$) as telling us how to use a point of X to define a path which stands still at that point (resp. how to use a path to define a map from a triangle which is constant in one direction, resp. how to use a morphism from a triangle to define a map from a tetrahedon which is constant in one direction).

Indeed, we saw that given any topological X, we can associate to it its *singular simplicial set* $Sing_\bullet X$ whose set of q-simplices is the set of maps

$$Sing_q X \stackrel{def}{=} \hom(\Delta^q_{top}, X)$$

where

$$\Delta^q_{top} = \{(t_0, \dots, t_q) : 0 \leq t_i \leq q, \sum t_i = 1\} \subseteq \mathbb{R}^{q+1}.$$

The face morphisms $d_i : Sing_q X \to Sing_{q-1} X$ were defined to be composition $f \mapsto f \circ |\delta_i|$ with the inclusions of the ith face $|\delta_i| : \Delta^{q-1}_{top} \subseteq \Delta^q_{top}; (t_0, \dots, t_{q-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{q-1})$, and the degeneracy morphisms $s_i : Sing_q X \to Sing_{q+1} X$ were defined to be composition $f \mapsto f \circ |\sigma_i|$ with the projections $|\sigma_i| : \Delta^{q+1}_{top} \to \Delta^q_{top}; (t_0, \dots, t_{q+1}) \mapsto (t_0, \dots, t_i + t_{i+1}, \dots, t_q)$ which send the ith and $(i+1)$th verticies to the same point.

Later in this talk, we will see that we will build a topological space from any simplicial set, and in the case of $Sing_\bullet X$ this reconstructs the space X (up to homotopy).

Notice that given a continuous morphism of topological spaces $X \to Y$, we obtain for every q a morphism of sets $Sing_q f : Sing_q X \to Sing_q Y$ by sending a map $\Delta^q_{top} \to X$ to the composition $\Delta^q_{top} \to X \to Y$. Moreover, these morphisms are compatible with the face morphisms d_i and the degeneracy morphisms s_i in the sense that

$$(Sing_{q-1} f) \circ d_i = d_i \circ (Sing_q f), \qquad (Sing_{q+1} f) \circ s_i = s_i \circ (Sing_q f)$$

for all q and i.

Definition 2. A *morphism* of simplicial sets $f : K \to L$ is a sequence of morphisms $f_q : K_q \to L_q$ such that

$$f_{q-1} d_i = d_i f_q, \qquad \text{and} \qquad f_{q+1} s_i = s_i f_q$$

for all i and q.

Example 3. Let $(P, \leq)$ be a partially ordered set. Recall that its nerve NP is the simplicial set whose set of q-simplicies is the set of q-tuples $(x_0, \dots, x_q) \in P^{q+1}$ such that $x_0 \leq \dots \leq x_q$. Face morphisms are induced by deleting an entry and degeneracy morphisms were induced by writing an entry twice.

Let $(Q, \leq)$ be another partially ordered set. and $f : (P, \leq) \to (Q, \leq)$ a morphism of partially ordered sets. That is, a morphism of sets $f : P \to Q$ such that for all $p, p' \in P$, if $p \leq p'$ then $f(p) \leq f(p')$. Then f induces a morphism of simplicial sets

$$Nf : NP \to NQ; \qquad (x_0, \dots, x_q) \mapsto (f(x_0), \dots, f(x_q)).$$

Example 4. Similarly, a morphism of directed graph induces a morphism of their associated nerves. We leave the details for the interested reader.

Recall that last week we also defined Δ^n to be the nerve of the partially ordered set $[n] = \{0, \dots, n\}$. So for example, $(0, 0, 1, 1, 1, 1, 3, 3)$ is an example of a 7-simplex of Δ^3.

We defined the product of two simplicial sets K, K' as the simplicial set whose set of q-simplicies are the set of pairs $(K \times K')_q = K_q \times K'_q$, and face and degeneracy morphisms are defined componentwise.

Definition 5. Let K, L be two simplicial sets. The *mapping space* from K to L is defined as the simplicial set whose set of q-simplices is the set of morphisms of simplicial sets from $\Delta^q \times K$ to L.

$$\mathrm{Map}(K, L)_q = \hom(K \times \Delta^q, L).$$

The face and degeneracy morphisms are defined as follows. For each i, there is a unique morphism of partially ordered sets $\delta_i : [q-1] \to [q]$ whose image is $\{0, 1, \dots, q\} \setminus \{i\}$. Similarly, there is a unique surjective morphism of partially ordered sets $\sigma_i : [q+1] \to [q]$ such that two elements of $[q+1]$ are sent to i. These morphisms of partially ordered sets induce morphisms of their nerves $N\delta_i : \Delta^{q-1} \to \Delta^q$ and $N\sigma_i : \Delta^{q+1} \to \Delta^q$ respectively, and precomposing with these morphisms, we obtain face morphisms

$$d_i : \hom(K \times \Delta^q, L) \to \hom(K \times \Delta^{q-1}, L); \qquad f \mapsto f \circ (\mathrm{id}_K \times N\delta_i)$$

and degeneracy morphisms

$$s_i : \hom(K\times\Delta^q, L) \to \hom(K\times\Delta^{q+1}, L); \qquad f \mapsto f \circ (\mathrm{id}_K \times N\sigma_i).$$

Last week we saw that the morphisms σ_i and δ_i satisfy the "opposite" properties to the axioms in Definition 1. From this one deduces that the definition of $\mathrm{Map}(K, L)$ satisfies the axioms in Definition 1.

II. Geometric realisation

Now let us consider how to reconstruct a topological space X from $Sing_\bullet X$. We recover the underlying set of X from $Sing_0 X = \hom(\Delta^0_{top}, X)$, since Δ^0_{top} is a single point, so a map from Δ^0_{top} to X is the same thing as choosing a point of X. So if K is a simplicial set we could start with K_0 as the set of points. We also know that for any element $\sigma \in Sing_1 X$ we have a path $\Delta^1_{top} \to X$ from the start $d_0\sigma$ to the end $d_1\sigma$. So in general, for every $\sigma \in K_1$, we want the geometric realisation $|K|$ to have a path from $d_0\sigma$ to $d_1\sigma$. Similarly, for every $\sigma \in K_2$ we want the geometric realisation to have a triangle, that is, a copy of Δ^2_{top}, whose edges are the paths $d_0\sigma, d_1\sigma$, and $d_2\sigma$.

Definition 6. Let K be a simplicial set. The geometric realisation $|K|$ of K is the topological space defined as follows. We begin with the disjoint union

$$\left(\coprod_{q\in\mathbb{N}} \coprod_{\alpha\in K_q} \Delta^q_{top}\right) = \{(q, \alpha, t) : q \in \mathbb{N}, \alpha \in K_q, t \in \Delta^q_{top}\}. \tag{1}$$

That is, for every q-simplex $\alpha \in K_q$ we have a copy of the topological q-simplex Δ^q_{top}. In general, this disjoint union will be enormous. Then we define an equivalence relation on this huge disjoint union of Δ_{top}'s. It is generated by the following relations. For every q, i we say that

$$(q, \alpha, |\delta_i|(t)) \sim (q-1, d_i\alpha, t), \qquad \text{and} \qquad (q, \alpha, |\sigma_i|(t)) \sim (q+1, s_i\alpha, t). \tag{2}$$

Here, $|\delta_i| : \Delta^{q-1}_{top} \to \Delta^q_{top}$ and $|\sigma_i| : \Delta^{q+1}_{top} \to \Delta^q_{top}$ are the morphisms induced the morphims $\delta_i : [q-1] \to [q]$ and $\sigma_i : [q+1] \to [q]$ of partially ordered sets. Explicitly, $|\delta_i|(t_0, \dots, t_{q-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{q-1})$ and $|\sigma_i|(t_0, \dots, t_{q+1}) = (t_0, \dots, t_i + t_{i+1}, \dots, t_{q+1})$.

We equip the quotient set $|K|$ with the quotient topology.

Let us investigate a bit closer what happens in low degrees in this construction. We start with a bunch of points, one for every element of K_0, a bunch of closed unit intervals, one for every element of K_1, a bunch of triangles, one for every element of K_2, etc etc. Now the relations. The relation corresponding to the degeneracy map $s_0 : [1] \to [0]$ means that for every interval corresponding to some element of the form $s_0\sigma \in K_1$ all its points are equivalent to the point $(0, \sigma, 1)$ corresponding to σ. On the other hand, for a general $\sigma \in K_1$, the endpoints $(1, \sigma, (1, 0))$ and $(1, \sigma, (0, 1))$ of its corresponding closed unit interval are associated to the points $(0, d_1\sigma, 1)$ and $(0, d_0\sigma, 1)$ corresponding to $d_0\sigma, d_1\sigma \in K_0$. In geometric terms, we have joined these two points by an interval. Similarly, the triangle corresponding to some $\sigma \in K_2$ will have its edges connected to the three closed unit intervals corresponding to $d_0\sigma, d_1\sigma, d_2\sigma$. Beyond this we can't say that much in general, its just an abstract topological space that we have constructed.

Lemma 7. *The geometric realisation $|\Delta^n|$ of the nth standard simplicial set Δ^n is homeomorphic to Δ^n_{top}.*

Proof. We have a map $\Delta^n_{top} \to |\Delta^n|$ defined by sending $t \in \Delta^n_{top}$ to $(n, (0, 1, \ldots, n), t)$. That is, we identify it with the copy of Δ^n_{top} in the disjoint union of Equation (1) corresponding to the n-simplex $(0, 1, \ldots, n) \in (\Delta^n)_n$. On the other hand, any q-simplex $(i_0, \ldots, i_q) \in (\Delta^n)_q$ defines a map $\alpha : [q] \to [n]$ (sending $j \in [q]$ to $i_j \in [n]$). This map induces a morphism of topological spaces $|\alpha| : \Delta^q_{top} \to \Delta^n_{top}$ sending $t_0 e_0 + \ldots t_q e_q$ to $t_0 e_{\alpha(0)} + \ldots t_q e_{\alpha(q)}$ where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{q+1}$ is the ith standard basis vector and $0 \leq t_i \leq 1, \sum t_i = 1$ are the coordinates of a point in Δ^q_{top}. Putting all of these $|\alpha|$ together, we get a map $\coprod_{q \in \mathbb{N}} \coprod_{\alpha \in K_q} \Delta^q_{top} \to \Delta^n_{top}$ which we claim is compatible with the equivalence relation, and therefore defines a map $|\Delta^n| \to \Delta^n_{top}$.

We claim that these two morphisms $\Delta^n_{top} \to |\Delta^n|$ and $|\Delta^n| \to \Delta^n_{top}$ are inverse homeomorphisms. Since $(0, 1, \ldots, n) \in (\Delta^n)_n$ corresponds to the identity morphism $[n] \to [n]$, and therefore the identity morphism $\Delta^n_{top} \to \Delta^n_{top}$, it follows that the composition $\Delta^n_{top} \to |\Delta^n| \to \Delta^n_{top}$ is the identity.

Therefore, modulo our claim that the $|\alpha|$ are compatible with the equivalence relation (i.e., that $|\Delta^n| \to \Delta^n_{top}$ is well-defined) it suffices to prove that $|\Delta^n| \to \Delta^n_{top}$ is surjective. In other words, it suffices to prove that every element (q, α, t) in the big disjoint union of Equation 2 is equivalent to some element of the form $(n, (0, 1, \ldots, n), t')$.

Consider some $(i_0, \ldots, i_q) \in (\Delta^n)_q$. One can show that there is a sequence of morphisms of the form

$$[q] \stackrel{\sigma j_1}{\to} [q-1] \stackrel{\sigma_{j_1}}{\to} \cdots \stackrel{\sigma_{j_{q-p}}}{\to} [p] \stackrel{\delta_{j_1}}{\to} [p+1] \stackrel{\delta_{j_2}}{\to} \cdots \stackrel{\delta_{j_{n-p}}}{\to} [n]$$

such that the composition is the morphism $\alpha : [q] \to [n]; j \mapsto i_j$. Via the equivalence relations of Equation 2, we then find that the point $t_0 e_0 + \ldots t_q e_q$ in the Δ^q_{top} corresponding to $(i_0, \ldots, i_q)$ is made to be equivalent to the point $t_0 e_{i_0} + \ldots t_q e_{i_q}$ in our distinguished copy of Δ^n_{top}, so $|\Delta^n| \to \Delta^n_{top}$ is surjective. □

III. Homotopy equivalence

Although $|Sing_\bullet X|$ is not homeomorphic to X in general, if X is not to pathological, the two spaces are *homotopic*. We define now what this means.

Definition 8. Let $f, g : X \to Y$ be two morphisms between two topological spaces. A *homotopy* from f to g is a morphism $h : X \times \Delta^1_{top} \to Y$ such that $h(x, (1, 0)) = f(x)$ and $h(x, (0, 1)) = g(x)$ for all $x \in X$. If there exists a homotopy from f to g we say that f and g are *homotopic*.

Given a homotopy as above, we may think of h as a continuous deformation of f to g over time, whose value at time $t \in [0, 1]$ is the map $h((t, 1-t), \bullet) : X \to Y$. Using this notion, we can make precise what it means to continuously deform one space into another space.

Definition 9. Let X, Y be topological spaces. If there exist morphisms $f : X \to Y$ and $g : Y \to X$ such that fg is homotopic to id_Y and gf is homotopic to id_X then we say that X and Y are *homotopic*, and f is a homotopy equivalence from X to Y.

Example 10. If $f : X \to Y$ is a homeomorphism with inverse f^{-1}, then it is a homotopy equivalence since $ff^{-1} = \mathrm{id}_Y$ and $f^{-1}f = \mathrm{id}_X$ so we can take the constant homotopy $h : X \times \Delta^1_{top} \to X; h(x,t) = x$ and similar for Y.

Example 11. Let $X = \{(x_0, x_1) \in \mathbb{R}^2 : x_0^2 + x_1^2 = 1\}$ and $Y = \mathbb{R}^2 \setminus \{(0,0)\}$. We claim that X and Y are homotopic. Define $f : X \to Y$ to be the inclusion, and $Y \to X$ to be the map $(x_0, x_1) \mapsto (\frac{x_0}{\sqrt{(x_0^2+x_0^2)}}, \frac{x_1}{\sqrt{(x_0^2+x_1^2)}})$. Then we have $g \circ f = \mathrm{id}_X$ so we can use the constant homotopy here. On the other hand, $f \circ g \neq \mathrm{id}_Y$. However, define $h : Y \times \Delta^1_{top} \to Y; ((y_0, y_1), (t_0, t_1) \mapsto (\frac{y_0}{t_1 + t_0\sqrt{(y_0^2+y_1^2)}}, \frac{y_1}{t_1+t_0\sqrt{(y_0^2+y_1^2)}})$. This gives a homotopy from id_Y to $f \circ g$.

A similar formula shows that in general, S^n is homotopic to $\mathbb{R}^{n+1} \setminus \{(0, \dots, 0)\}$.

Example 12. We claim that each Δ^n_{top} is homotopic to a point, say $e_0 = (1, 0, \dots, 0)$. If f is the inclusion $e_0 \in \Delta^n_{top}$, and g is the projection sending every element of Δ^n_{top} to e_0, then $gf = \mathrm{id}$, and $fg \neq \mathrm{id}$, but we can define a homotopy $h : \Delta^n_{top} \times \Delta^1_{top} \to \Delta^n_{top}$ by $((x_0, \dots, x_n), (t_0, t_1)) \mapsto (t_1 + t_0x_0, t_0x_1, \dots, t_0x_n)$. This gives a homotopy from fg to id.

Example 13. There is always a canonical morphism $|Sing_\bullet X| \to X$ (see if you can guess the definition, it is not hard, see the proof of Lemma 7 for a clue). If X is a "nice" topological space (for example a CW complex) then this canonical morphism is a homotopy equivalence.

Definition 14. We say that a morphism $f : K \to L$ of simplicial sets is a *weak equivalence* if the geometric realisation $|f| : |K| \to |L|$ is a homotopy equivalence.

Remark 15. There is a combinatorial way to define weak equivalences of simplicial sets without using topological spaces. First one defines the homotopy groups of a Kan complex (Kan complexes will be defined in the next section; their homotopy groups are defined in [May, Def.3.6] and [Wei, Def.8.3.1] but beware that their notation is slightly different to ours). Then one says that a morphism of Kan complexes is a weak equivalence if it induces an isomorphism on all homotopy groups. For morphisms between simplicial sets K, L which are not Kan complexes, we take canonical weak equivalences $K \to RK$, $L \to RL$ where RK, RL are Kan complexes, and then define $K \to L$ to be a weak equivalence if its induced morphism $RK \to RL$ induces an isomorphism an homotopy groups. For example, there is a way to define barycentric subdivision of simplicial sets, and using this, one associates to each simplicial set K a Kan complex $Ex^\infty K$ and to each morphism of simplicial sets $f : K \to L$ a morphism of Kan complexes $Ex^\infty f : Ex^\infty K \to Ex^\infty L$. Another option is the canonical morphisms $K \to Sing_\bullet|K|$, $L \to Sing_\bullet|L|$, although this doesn't avoid using the geometric realisation.

IV. Kan complexes

Recall that last week we saw that the subtopological space of Δ^2_{top} corresponding to the simplicial set $N\{0,1\} \cup N\{0,2\} \cup N\{1,2\}$ was its boundary. This is true for higher n too, and leads to the following definition.

Definition 16. The boundary of Δ^n is the subsimplicial set

$$\partial\Delta^n \overset{def}{=} \bigcup_{i=0}^{n} N\Big(\{0,\dots,n\}\setminus\{i\}\Big).$$

We also saw other subspaces of Δ^2_{top} where we only took the union of two of the $N\{0,1\}, N\{0,2\}, N\{1,2\}$. This generalises to higher dimensions too. We define the kth horn as the boundary with the kth face missing as follows.

Definition 17. The kth horn of Δ^n is the subsimplicial set

$$\Lambda^n_k \overset{def}{=} \bigcup_{i\neq k} N\Big(\{0,\dots,n\}\setminus\{i\}\Big).$$

Example 18. The set of 7-simplicies of the simplicial set Δ^5 is the set of tuples $(i_0,i_1,i_2,i_3,i_4,i_5,i_6,i_7)$ with $0\le i_0\le\cdots\le i_7\le 5$, for example $(0,1,1,2,2,4,5,5)$ or $(0,1,1,1,1,1,2,3)$. The 7-simplicies contained in $\partial\Delta^5$ are then those simplicies which don't contain every element of [5]. So, the two examples above are ok, but $(0,1,2,3,3,4,4,5)$ is in Δ^5 but not $\partial\Delta^5$. For the 7-simplicies of Λ^5_3, we are still not allowed to have every element of [5], but moreover, we are also not allowed to have every element of $\{0,1,2,4,5\}$, so $(0,1,1,2,2,4,5,5)$ is not in Λ^5_3 but $(0,1,1,1,1,1,2,3)$ is in Λ^5_3.

Now we define Kan complexes. These are the simplicial sets that satisfy a "horn-filling" property—every time we have a bunch of q-simplicies whose various faces agree so that they fit together into the shape of a horn (in the above sense), there exists a $(q+1)$-simplex for which these q-simplices are its faces. More precisely:

Definition 19. Let K be a simplicial set. We say K satisfies the (n,i)-*lifting property* if for every set of simplicies $\alpha_0,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n\in K_{n-1}$ such that

$$d_j\alpha_k = d_{k-1}\alpha_j,\qquad j<k,\qquad j,k\neq i, \tag{3}$$

there exists $\beta\in K_n$ such that

$$d_j\beta=\alpha_j,\qquad j\neq i. \tag{4}$$

Equivalently, K satisfies the (n,i)-lifting property if any morphism of the form $\Lambda^n_i\to K$, can be extended to some morphism $\Delta^n\to K$.

Definition 20. We say that K is a *Kan complex* if it satisfies the (n,i)-lifting property for all $n\in\mathbb{N}$ and all $0\le i\le n$.

Proposition 21. *For any topological space X, the simplicial set $Sing_\bullet X$ is a Kan complex.*

Sketch of proof. The idea is that a morphism of simplicial sets $\alpha:\Lambda^n_i\to Sing_\bullet X$ (resp. $\beta:\Delta^n\to Sing_\bullet X$) is the same thing as a continuous morphism $a:|\Lambda^n_i|\to X$ (resp. $b:\Delta^n_{top}$). Then, since for every inclusion $|\Lambda^n_i|\subset\Delta^n_{top}$ we can find a continuous map $r:\Delta^n_{top}\to|\Lambda^n_i|$ which is the identity when restricted to $|\Lambda^n_i|$, given any a as above, we can define b as the composition $a\circ r:\Delta^n_{top}\to|\Lambda^n_i|\to Sing_\bullet X$. Note that since the retraction r is not unique, the lifting property does not have a unique solution—in general there are many options! $\square$

Example 22. Let G be a group, and consider its classifying space BG. Recall that the q-simplicies are tuples $(g_1, \ldots, g_q)$. There is a unique 0-simplex—the empty tuple (). Degeneracy morphisms s_i insert the identity element $e \in G$ in position i. Face morphisms d_0 (resp. d_n) cancel the first (resp. last) coordinate, and the other d_i are $(g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_i g_{i+1}, \ldots, g_n)$.

We claim that BG is a Kan complex. Just to give an idea of the proof, we show the (n, i)-lifting property for $(n, i) = (2, 0), (2, 1), (2, 2)$. The argument for higher n, i is the same, but with more bookkeeping.

Case (2,0). We have $\alpha_1 = g_1$ and $\alpha_2 = g_2$ and the condition of Equation 3 is trivially satisfied because the set BG_0 has a single element. Defining $\beta = (g_2, g_2^{-1} g_1)$ we check that $d_1\beta = g_2 g_2^{-1} g_1 = g_1$ and $d_2\beta = g_2$.

Case (2, 1). We have $\alpha_0 = g_0$ and $\alpha_2 = g_2$. Define $\beta = (g_2, g_0)$ and check that $d_2\beta = g_2$ and $d_0\beta = g_1$.

Case (2, 2). We have $\alpha_0 = g_0$ and $\alpha_1 = g_1$. Define $\beta = (g_1 g_0^{-1}, g_0)$ and check that $d_0\beta = g_0$ and $d_1\beta = g_1 g_0^{-1} g_0 = g_1$.

Later on we will need a more general notion—the notion of a Kan fibration. Heuristically, this is a morphism of simplicial sets for which all of the fibres are Kan complexes.

Definition 23. Let $K \to L$ be a morphism of simplicial sets. We say that f is a *Kan fibration* if for every $n \in \mathbb{N}$, every $0 \leq i \leq n$, and every commutative square as below, there exists a diagonal morphism making the two triangles commute.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & K \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & L. \end{tikzcd}$$

This condition can also be expressed in terms of α's and β's as in Definition 19, but we leave this reformulation as an exercise for the motivated reader.

Notice that K is a Kan complex if and only if the canonical, unique morphism to Δ^0 is a Kan fibration.

V. ∞-CATEGORIES

Definition 24. An ∞-*category* is a simplicial set K satisfying the (n, i)-lifting property of Definition 19 for all $n \in \mathbb{N}$, and $0 < i < n$.

Remark 25. Note that any Kan complex is an ∞-category. In particular, the $Sing_\bullet X$ are ∞-categories.

Example 26. Let $(P, \leq)$ be a partially ordered set. We claim that NP is an ∞-category. Suppose that we are given $\alpha_j = (x_{j,0}, \ldots, x_{j,n-1}) \in P^n = (NP)_{n-1}$ satisfying the condition of Equation 3. Write these in an $(n+1) \times (n+1)$ matrix, by inserting an $*$ on the diagonal entries, and $*$'s in the ith row.

$$\begin{pmatrix} * & x_{0,0} & x_{0,1} & \dots & x_{0,n-1} \\ x_{1,0} & * & x_{1,1} & \dots & x_{1,n-1} \\ x_{2,0} & x_{2,1} & * & \dots & x_{1,n-1} \\ \vdots & & & & \vdots \\ * & * & * & * & * \\ \vdots & & & & \vdots \\ x_{n,0} & x_{n,1} & \dots & x_{n,n-1} & * \end{pmatrix}$$

Since the face morphisms d_i of NP are defined by omitting an element, the (n, i)-lifting condition of Equation 19 shows that all the columns of this matrix are the same. Indeed, it says that for $j < k$ with $j, k \neq i$, removing the jth element from the kth row, and the kth element from the jth row, these two rows become equal. So now just define β to be the tuple $(y_0, \dots, y_n)$ where y_j is the entry in the jth column of the matrix. It is straightforward now to see that the condition of Equation 4 is satisfied. We remark that we can also observe that the β we obtain is ALWAYS unique in this example.

Example 27. Let $\mathcal{G} = (V, E)$ be a directed graph. We claim that $N\mathcal{G}$ is an ∞-category. We use the same trick as for the partially ordered set. Suppose we are given $\alpha_j = (\alpha_{j,1}, \dots, \alpha_{j,n-1}) \in N\mathcal{G}_{n-1}$ satisfying the condition of Equation 3, where each $\alpha_{j,k}$ is a path of edges through our directed graph. Write them in a matrix as we did for the partially ordered set, this time the matrix is a $(n+1) \times n$-matrix, and so both of the last two rows have an $*$ in the last column.

$$\begin{pmatrix} * & \alpha_{0,1} & \alpha_{0,2} & \dots & \alpha_{0,n-1} \\ \alpha_{1,1} & * & \alpha_{1,2} & \dots & \alpha_{1,n-1} \\ \alpha_{2,1} & \alpha_{2,2} & * & \dots & \alpha_{1,n-1} \\ \vdots & & & & \vdots \\ * & * & * & * & * \\ \vdots & & & & \vdots \\ \alpha_{n-1,1} & \alpha_{n-1,2} & \dots & \alpha_{n-1,n-1} & * \\ \alpha_{n,1} & \alpha_{n,2} & \dots & \alpha_{n,n-1} & * \end{pmatrix}$$

If $j \neq 0, n-1$, then when $j < k$, applying d_j to the row with α_k in it replaces the entry $\alpha_{k,j}$ with the concatenation $\alpha_{k,j}\alpha_{k,j+1}$, and puts a $*$ where $\alpha_{k,j+1}$ was. Applying d_{k-1} to the row with α_j in it replaces the entry $\alpha_{j,k-1}$ with the concatenation $\alpha_{j,k-1}\alpha_{j,k}$ and puts a $*$ where $\alpha_{j,k}$ was. Hence, the (n, i)-lifting condition of Equation 19 shows that all the columns of this matrix are the same EXCEPT for the entries $\alpha_{j,j}$ just left of the diagonal. Here the lifting condition shows that $\alpha_{j,j} = \alpha_{k,j}\alpha_{k,j+1}$. In other words, there are paths $\beta_1, \dots, \beta_n$ such that our matrix looks like this:

$$\begin{pmatrix} * & \beta_2 & \beta_3 & \dots & \beta_n \\ \beta_1\beta_2 & * & \beta_3 & \dots & \beta_n \\ \beta_1 & \beta_2\beta_3 & * & \dots & \beta_n \\ \vdots & & & & \vdots \\ * & * & * & * & * \\ \vdots & & & & \vdots \\ \beta_1 & \beta_2 & \dots & \beta_{n-1}\beta_n & * \\ \beta_1 & \beta_2 & \dots & \beta_{n-1} & * \end{pmatrix}$$

Then one sees immediately that defining β_j to be any of the equal entries in the jth column (that is, not $\alpha_{j,j}$), provides the unique solution when $n > 1$.

Definition 28. Let C be an ∞-category. We call C an *n-category* if for every $m > n$ and $0 < i < m$, the β in the definition of the (m, i)-lifting property is unique. Equivalently, the extension $\Delta^m \to C$ in the definition of the (m, i)-lifting property is unique.

Example 29. Recall, that we have already observed the uniqueness for nerves of partially ordered sets. That is, we observed that they are 0-categories. In fact, every 0-category corresponds to a unique partially ordered set, and conversely.

Example 30. We noted above that the ∞-category $Sing_\bullet X$ is not an n-category for any n (unless of course X is a discrete set of points, or empty).

Definition 31. Traditionally, a *category* C was defined to be something like a directed graph with a multiplication operation on edges. More precisely, a category is:

(1) A set $Ob\ C$ whose elements are called *objects*.
(2) For each pair of objects $x, y \in Ob\ C$ a set $\hom(x, y)$ whose elements are called *morphisms*.
(3) For each triple of objects x, y, z a morphism of sets
$$\circ : \hom(y, z) \times \hom(x, y) \to \hom(x, z)$$
called *composition*.

This data is required to satisfy:

(1) For each object x there is an element $\mathrm{id}_x \in \hom(x, x)$ which satisfies $\mathrm{id}_x \circ f = f$ and $g \circ \mathrm{id}_x = g$ for every $y \in Ob\ C$, every $f \in \hom(y, x)$ and every $g \in \hom(x, y)$.
(2) For objects w, x, y, z and morphisms $f \in \hom(w, x), g \in \hom(x, y), h \in \hom(y, z)$ we have $(h \circ g) \circ f = h \circ (g, \circ f)$.

Examples of categories are the category of groups (resp. vector spaces, resp. topological spaces, resp. sets) whose objects are the collection of all groups (resp. vector spaces, resp. topological spaces, resp. sets) and for any two objects x, y, the set $\hom(x, y)$ is the set of group homomorphims (resp. linear maps, resp. continuous morphisms, resp. morphism of sets). The motivation was that there were many procedures, such as the fundamental group $\pi_1 X$ of a topological space X, which associated an object (resp. morphism) in one category to an object (resp. morphism) in another category, in such a way that the composition of two morphisms was sent to the composition of their images.

Definition 32. Given a category C, as defined above, we can associate to it a simplicial set NC called the *nerve*. The 0-simplicies are the objects of C, the 1-simplicies are all the morphisms (for all pairs of objects), $d_0, d_1 : NC_1 \to NC_0$ send a morphism $f \in \hom(x, y)$ to its *source* x and *target* y respectively, $s_0 : NC_0 \to NC_1$ sends an object x to its identity morphism id_x. More generally, the set NC_q of q-simplicies is the set of tuples of composable morphisms, i.e., tuples $(f_0, \dots, f_q)$ such that the target of f_i is the source of f_{i+1}, the boundary morphisms are defined using composition as in the classifying space BG of a group G, and the degeneracy morphisms are defined by inserting an identity morphism (as in the classifying space BG of a group G).

Example 33. Indeed, given a group G, one can define a category which has one object, say called $*$, and $\hom(*, *) = G$. Then BG is precisely the nerve of this category.

Example 34. Given a partially ordered set $(P, \leq)$, one can define a category whose objects are the elements of P, and $\hom(x, y)$ has a unique morphism if $x \leq y$, and is empty otherwise. Then the nerve of this category is precisely the nerve of the partially ordered set.

Example 35. Given a directed graph, one can associate a category to it called the *free category*. Its object are the nodes of the graph, and the morphisms are sequences of edges $e_1, \ldots, e_n$ such that the source of e_i is the target of e_{i-1}. Composition of morphisms We formally include an empty path for every node which correspond to the identity morphisms. Then the nerve of this free category is precisely the nerve of the directed graph.

Remark 36. Note that we can reconstruct any category C, from its nerve NC. Conversely, one can show that any 1-category is the nerve of a unique category. That is, categories in the older sense are the same thing as 1-categories in our terminology.

VI. Functors

Definition 37. Let C, C' be ∞-categories. A *functor* is just a morphism of simplicial sets $C \to D$. A *natural transformation* from a functor f to a functor g is a morphism of simplicial sets $\eta : C \times \Delta^1 \to D$ such that $\eta|_{C \times N\{0\}} = f$ and $\eta|_{C \times N\{1\}} = g$ (note that there is a canonical isomorphism $C \times N\{*\} \cong C$ for any partially ordered set $\{*\}$ with one element, such as $\{0\}$ or $\{1\}$).

Proposition 38. *If D is an ∞-category and C a simplicial set then the mapping space* $\mathrm{Map}(C, D)$ *is an ∞-category.*

Due to this proposition, when C and D are ∞-categories, and we want to think of $\mathrm{Map}(C, D)$ as an ∞-category we will use the notation

$$\mathrm{Fun}(C, D) = \mathrm{Map}(C, D).$$