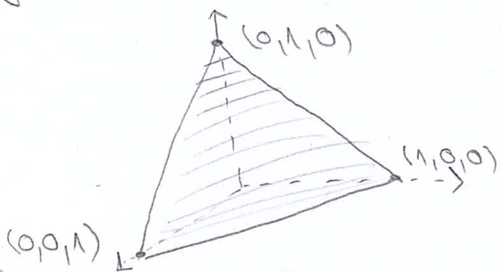
Simplicial sets

DEF. The topological n simplex $\Delta^n_{top} := \{(t_0,\dots,t_n) \in \mathbb{R}^{n+1} \mid 0 \le t_i \le 1, \sum_{i=0}^{n} t_i = 1\}$

~ We define face morphisms $\delta_i : \Delta^{n-1}_{top} \to \Delta^n_{top}$ $\delta_i(t_0,\dots,t_{n-1}) = (t_0,\dots,t_{i-1},0,t_i,\dots,t_{n-1})$
degeneracy morphisms $\sigma_i : \Delta^{n+1}_{top} \to \Delta^n_{top}$ $\sigma_i(t_0,\dots,t_{n+1}) = (t_0,\dots,t_i+t_{i+1},\dots,t_{n+1})$

EXA. of Δ^n_{top}

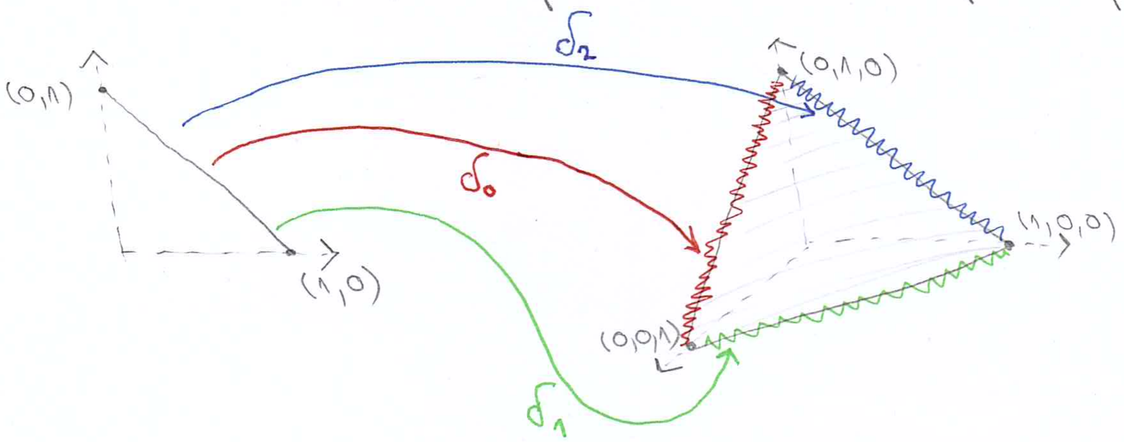
1. $n=0$ we get just a point $1 \in \mathbb{R}$
2. $n=1$ we get an interval between $(0,1)$ and $(1,0) \in \mathbb{R}^2$
3. $n=2$



4. $n=3$ tetrahedron in $\mathbb{R}^4$

EXA. of morphisms $\delta_0, \delta_1, \delta_2 : \Delta^1_{top} \to \Delta^2_{top}$ and $\sigma_0, \sigma_1 : \Delta^2_{top} \to \Delta^1_{top}$

1) $\delta_1, \delta_2, \delta_3$

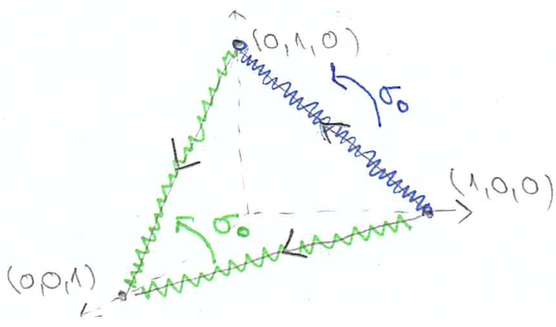


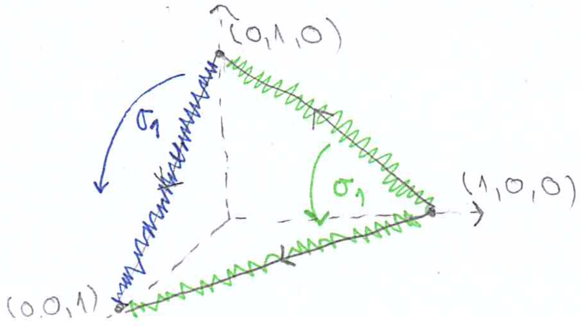
2) σ_0

$\sigma_0(1,0,0) = (1,0) = \sigma_0(0,1,0)$
$\sigma_0(0,0,1) = (0,1)$

σ_1

$\sigma_1(1,0,0) = (1,0)$
$\sigma_1(0,1,0) = \sigma_1(0,0,1) = (0,1)$





DEF. Given a topological space X, one can associate to it
$\mathrm{Sing}_n(X) :=$ set of continuous maps from Δ_n^{top} to X.

Remark: We can use $\mathrm{Sing}\, X = \{\mathrm{Sing}_n(X)\}$ to construct a top. space that is (weakly) homotopy equivalent to the original space.

Exercise: Show that face and degeneracy morphisms induce morphisms $d_i : \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n-1}(X)$, $s_i : \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n+1}(X)$
such that: i) $d_i d_j = d_{j-1} d_i$ if $i<j$
i) $s_i s_j = s_{j+1} s_i$ if $i \leq j$
iii) $d_i s_j = s_{j-1} d_i$ if $i<j$
$d_j s_j = \text{identity} = d_{j+1} s_j$
$d_i s_j = s_j d_{i-1}$ if $i>j$

1 Show it holds for δ_i, σ_i $\quad \delta_i : \Delta^{n-1}_{top} \to \Delta^n_{top}$, $\sigma_i : \Delta^{n+1}_{top} \to \Delta^n_{top}$

i) $\sigma_i \sigma_j = \sigma_{j-1} \sigma_i \quad i<j$

$$\begin{aligned}\sigma_i \sigma_j (t_0, \dots, t_{n+1}) &= \sigma_i (t_0, \dots, t_j + t_{j+1}, \dots, t_{n+1}) \\ &= (t_0, \dots, t_i + t_{i+1}, \dots, t_j + t_{j+1}, \dots, t_{n+1}) \\ &= \sigma_{j-1}(t_0, \dots, t_i + t_{i+1}, \dots, t_{n+1}) \\ &= \sigma_{j-1}\sigma_i(t_0, \dots, t_{n+1})\end{aligned}$$

ii) $\delta_i \delta_j = \delta_{j+1} \delta_i$ if $i \leq j$

$$\begin{aligned}\delta_i \delta_j (t_0, \dots, t_{n-1}) &= \delta_i (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}) \\ &= (t_0, \dots, t_{i-1}, 0, t_i, \dots t_{j-1}, 0, t_j, t_{n-1}) \\ &= \delta_{j+1}(t_0, \dots, t_{i-1}, 0, t_i, t_{n-1}) = \delta_{j+1}\delta_i(t_0, \dots, t_{n-1})\end{aligned}$$

2 Show it holds for s_i, d_i

i) $d_i : \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n-1}(X)$
$(f : \Delta^n_{top} \to X) \to (f \circ \delta_i : \Delta^{top}_{n-1} \to X)$

ii) $s_i : \mathrm{Sing}_n(X) \to \mathrm{Sing}_{n+1}(X)$
$(f : \Delta^{top}_n \to X) \to (f \circ \sigma_i : \Delta^{top}_{n+1} \to X)$

$d_i d_j(f) = d_i(d_j(f)) = d_i(f \circ \delta_j) = f \circ \delta_j \circ \delta_i$
$= f \circ \delta_i \circ \delta_{j-1} = d_{j-1} d_i(f)$

$s_i s_j(f) = f \circ \sigma_j \circ \sigma_i =$
$= f \circ \sigma_i \circ \sigma_{j+1}$
$= s_{j+1} s_i(f)$

DEF. A *simplicial set* K is a sequence of sets $K_0, K_1, K_2, \dots$, together with maps $d_i: K_q \to K_{q-1}$ and $s_i: K_q \to K_{q+1}$, $0 \le i \le q$ which satisfies:

$$\left.\begin{array}{ll} \text{i)} & d_i d_j = d_{j-1} d_i \quad \text{if } i<j \\ \text{ii)} & s_i s_j = s_{j+1} s_i \quad \text{if } i \le j \\ \text{iii)} & d_i s_j = s_{j-1} d_i \quad \text{if } i<j \\ & d_j s_j = \text{identity} = d_{j+1} s_j \\ & d_i s_j = s_j d_{i-1} \quad \text{if } i > j+1 \end{array}\right\} *$$

The elements of K_q are called *q simplices*. The maps d_i and s_i are called *face and degeneracy operators.*

Note: $\operatorname{Sing} X = \{\operatorname{Sing}_n(X)\}$ has the structure of simplicial set as we proved above.

Interpret.: We have a set K_q for every $0 \le q < \infty$ which we can think of as maps from an q-dim simplex into a space, and various morphisms d_i and s_i telling us how the triangles fit together

DEF. If $(P, \le)$ is a partially ordered set then we define a *simplicial set* NP (N is for "nerve") as follows:

$NP_q = \{(x_0, \dots, x_q) \in P^{q+1} \mid x_0 \le x_1 \le \dots \le x_q\}$ together with the maps
$\Rightarrow$ the set of q-simplices

$d_i: NP_q \to NP_{q-1}$, $\quad (x_0, \dots, x_q) \to (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_q)$

$s_i: NP_q \to NP_{q+1}$, $\quad (x_0, \dots, x_q) \to (x_0, \dots, x_i, x_i, \dots, x_q)$

PROOF that the maps agree with $*$

i) $d_i d_j (x_0, \dots, x_q) = d_i (x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_q)$
$= (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_q)$
$= d_{j-1} (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_q)$
$= d_{j-1} d_i (x_0, \dots, x_q)$

ii) $s_i s_j (x_0, \dots, x_q) = s_i (x_0, \dots, x_j, x_j, \dots, x_q)$
$= (x_0, \dots, x_i, x_i, \dots, x_j, x_j, \dots, x_q) \qquad i \le j$
$= s_{j+1} (x_0, \dots, x_i, x_i, \dots, x_q)$
$= s_{j+1} s_i (x_0, \dots, x_q)$

DEF. A *0-category* is a simplicial set K that is a nerve of some partially ordered set $(X, \le)$.

$K = NP$

DEF. $(P,\leq)=[n]=\{0\leq 1\leq \dots \leq n\}$ is a partially ordered set. We define simplicial set Δ^n as follows:

$$\Delta^n_q=\{(x_0,\dots,x_q)\in [n]^{q+1} \mid x_0\leq x_1\leq \dots \leq x_n\}$$

$$d_i:\Delta^n_q\to\Delta^n_{q-1} \qquad (x_0,\dots,x_q)\to(x_0,\dots,x_{i-1},x_{i+1},\dots,x_q)$$

$$s:\Delta^n_q\to\Delta^n_{q+1} \qquad (x_0,\dots,x_q)\to(x_0,\dots,x_i,x_i,\dots x_q)$$

Directed graphs

- path in directed graph is sequence of edges $e_1,e_2,\dots,e_i$ where the target of e_i = source of e_{i+1}. For a directed graph G set of 0-simplicies is a set of vertices and the q-simplicies are q-tuples of paths:

$$NG_q=\{(\underbrace{e_{1,1}e_{1,2}\dots e_{1,i_1}}_{\text{path}},e_{2,1}e_{2,2}\dots e_{2,i_2},\dots,e_{q,1}e_{q,2}\dots e_{q,i_q}) \mid \text{s.t. source } e_{j,i_j}=\text{target } e_{j+1,1}\}$$

$d_i:NG_q\to NG_{q-1}$ concatenating i-th path with $i+1$ path (d_0,d_q remove first and last path)

$s_i:NG_q\to NG_{q+1}$ inserts empty path (no edges) in $(i+1)$-th position
(s_0,s_q put empty paths in the first and the last position)

DEF. (Classifying space) Let G be a group and consider the simplicial set BG defined by $BG_0=\{1\}$, $BG_1=G,\dots,BG_n=G^n$

$$s_i(g_1,\dots,g_n)=(g_1,\dots,g_i,1,g_{i+1},\dots,g_n)$$

$$d_i(g_1,\dots,g_n)=\begin{cases}(g_2,\dots,g_n) & \text{if } i=0\\ (g_1,\dots,g_{i-1},g_{i+1},\dots,g_n) & \text{if } 0<i<n\\ (g_1,\dots,g_{n-1}) & \text{if } i=n\end{cases}$$

Remark: Used when one wants to prove that the space is a loop space.

DEF. A **subsimplicial set** of a simplicial set K_* is a sequence of subsets $L_0 \subseteq K_0$, $L_1 \subseteq K_1$, $L_2 \subseteq K_2$ ... such that:
$d_i(L_q) \subseteq L_{q-1}$ and $s_i(L_q) \subseteq L_{q+1}$ for all $0 \leq i \leq q$, $q = 0,1,2,3,\ldots$

EXA. 1) Simplicial set Δ^3:
$(\Delta^3)_q = \{(x_0,\ldots,x_q) \mid 0 \leq x_0 \leq \ldots \leq x_q \leq 3\}$

$(\Delta^1)_q = \{(x_0,\ldots,x_q) \mid 0 \leq x_0 \leq \ldots \leq x_q \leq 1\}$

Δ^1 is subsimplicial set of set Δ^3

2) Whenever we have subset Q of partially ordered set P, applying the nerve operation we get a subsimplicial set NQ of a simplicial set NP

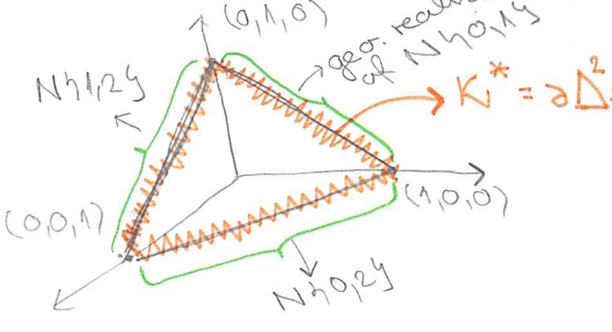
$I \subseteq \{0,1,2,3\}$
$NI \subseteq \Delta^3 := N\{0,1,2,3\}$

3) Connection between Δ^n_{top} and Δ^n
Notice that Δ^n_{top} has a „corner" for every e_i.
For every q-simplex $(x_0,\ldots,x_q)$ in Δ^n_q (that is sequence of $x_i \in \{0,1,\ldots,n\}$ such that $x_0 \leq \ldots \leq x_q$) we can consider the convex hull of points $e_{x_0},\ldots,e_{x_q}$
So now lets assign a top. subspace of Δ^n_{top} to a simplex in Δ^n

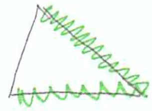
$$K \subseteq \Delta^n \longrightarrow K^* \subseteq \Delta^n_{top}$$
$$K^* = \bigcup_{q=0,1,\ldots} \bigcup_{\in\{(x_0,\ldots,x_q) \mid x_0 \leq \ldots x_q\}} \text{convex hull of } \{e_{x_0},\ldots,e_{x_q}\}$$
union of all subspaces of the simplaces of K

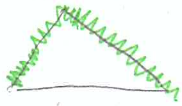
So for example $N\{0\} \subseteq \Delta^2$ we get $(1,0,0)$ in $\mathbb{R}^3$, $N\{1\} \to (0,1,0)$, $N\{2\} \to (0,0,1)$
Look now $N\{0,1\}, N\{0,2\}, N\{1,2\}$ where $(N\{0,1\})_q = \{(x_0,\ldots,x_q) \mid 0 \leq x_0 \ldots \leq x_q \leq 1\}$

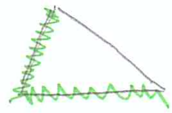
$$N\{0,1\} \cup N\{0,2\} \cup N\{1,2\} \subseteq \Delta^2 = N\{0,1,2\}$$



$$\Lambda^n_k = \bigcup_{i \neq k} N(\{0,\dots,n\} \setminus \{i\})$$

$\Lambda^2_0 = N\{0,1\} \cup N\{0,2\}$ 

$\Lambda^2_1 = N\{0,1\} \cup N\{1,2\}$ 

$\Lambda^2_2 = N\{0,2\} \cup N\{1,2\}$ 

DEF. Let K and K' be simplicial sets then the product simplicial set $K \times K'$ has as q-simplicies the set $(K \times K')_q = K_q \times K'_q$ of pairs (s,t) where $s \in K$ is a q-simplex of K ant $t \in K'$ is a q simplex of K'. The face and degeneracy maps are those of K and K' acting on each component of the pair retrospectively.

(exa. $d_i(s,t) = (d_i(s), d'_i(t))$)

Exercise: Show that for any 2 top. spaces X, Y:

$$\mathrm{Sing}(X \times Y) = \mathrm{Sing}(X) \times \mathrm{Sing}(Y)$$

fact: For any top. space Z we have $\hom(Z, X \times Y) = \hom(Z,X) \times \hom(Z,Y)$

$$\mathrm{Sing}_n(X \times Y) = \hom(\Delta^n_{top}, X \times Y) = \hom(\Delta^n_{top}, X) \times \hom(\Delta^n_{top}, Y) = \mathrm{Sing}_n(X) \times \mathrm{Sing}_n(Y)$$

$$\shortparallel$$

$$(\mathrm{Sing}\, X \times \mathrm{Sing}\, Y)_n$$

$d_i : \mathrm{Sing}_n \to \mathrm{Sing}_{n-1}$

$f \to f \circ \delta_i$

$$\begin{tikzcd} f \arrow[r, "{p,q}"] \arrow[d, "d_i"] & (p \circ f, q \circ f) \arrow[d] \\ f \circ \delta_i \arrow[r, "{p,q}"] & (p \circ f \circ \delta_i, q \circ f \circ \delta_i) \end{tikzcd}$$

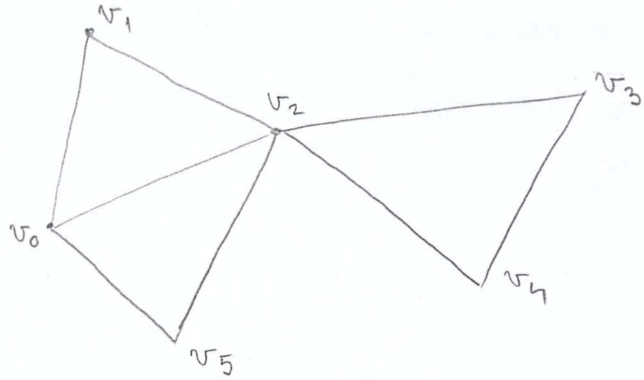
Where $p : X \times Y \to X$

$q : X \times Y \to Y$

DEF. A simplicial complex X in $\mathbb{R}^n$ consist of collection of simplices (n-simplex is convex set spanned by $n+1$ aff. independent points), possibly of various dimensions st:

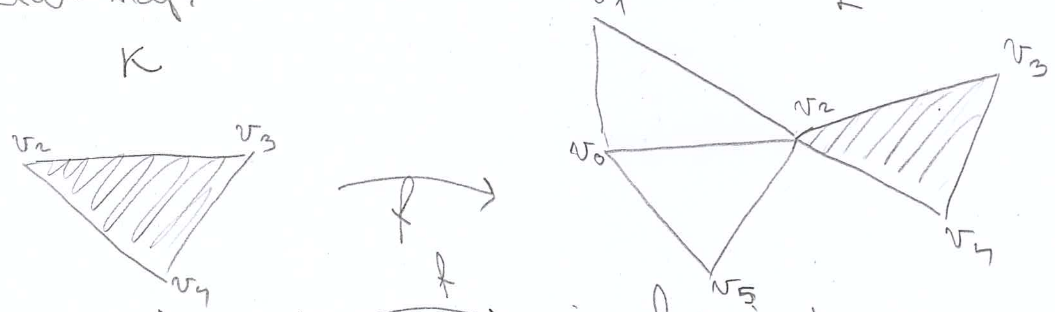
1) every face of a simplex of X is in X
2) the intersection of any 2 simplices in X is a face of each of them.

~ We can think of k-simplex as a simplicial complex consisting of itself and its faces.

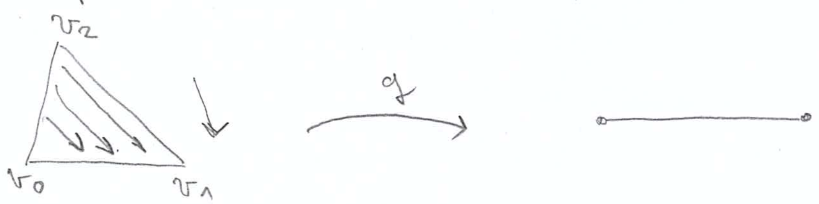


abstract simplicial complex ⇝ combinatorial info of a simplicial complex without geometry (embedding in Eucl. space)

simplicial map



simplex in $K \xrightarrow{f}$ simplex in L



Ordered simplicial complex → set of vertices totally ordered

$\mathrm{conv}(v_{i_0}, \dots, v_{i_k})$ = simplex iff $v_{i_j} < v_{i_l}$ $j < l$

Observe ordered simplicial complex → simplicial set

for every simplex $[v_{i_0}, \dots, v_{i_n}]$ → simplices of the form

ordered

$[v_{i_0}, \dots, v_{i_0}, v_{i_1}, \dots, v_{i_1}, \dots, v_{i_n}]$

for any number of repetit

simplicial set → ordered simplicial complex

(forget degeneracy maps)