∞-CATEGORIES SEMINAR

Contents

"Abstract"

Higher category theory lies in the intersection of two major developments of 20th century mathematics: topology and category theory. It provides a framework for settings where the morphisms between two objects form not just a set but a

topological space, and is believed to be the best language for modern homological algebra and sheaf theory by a steadily increasing portion of the mathematical community.

Despite being a new discipline, higher category theory has already found spectacular applications across mathematics, such as Lurie's proof of the cobordism hypothesis (in Mathematical Physics), and Gaitsgory-Lurie's work on Weil's Tamagawa number conjecture (in Number Theory), not to mention applications in Geometric Langlands, K-theory, Mirror Symmetry, Knot Theory / Floer Homology...

This reading seminar will gently introduce some of the main concepts of higher category theory as developped by Lurie. By the end of the seminar, the student will be familiar enough with infinity categories that they can navigate texts written in this new language.

General information

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Textbooks:	"A short course on ∞-categories" by Groth "Higher topos theory" by Lurie "Higher algebra" by Lurie See also the bibliography at the end.
Room:	SR 140/A7 Seminarraum (Hinterhaus) (Arnimallee 7)
Time:	Mo 16:00-18:00

About the presentation

This is a student seminar which means that the students each make one of the presentations. The presentation should be about 75 minutes long, leaving 15 minutes for potential questions and discussion.

Students are not *required* to hand in any written notes. However, students are encouraged to prepare some notes if they feel it will improve the presentation. This should be considered seriously, especially if the student has not made many presentations before.

For example, its helpful to have

(1) a written copy of exactly what they plan to write on the blackboard, and
(2) 5-10 pages of notes on the material to help find any gaps in your understanding.

If notes are prepared I will collect them and give feedback if desired.

The material listed below should be considered as a skeleton of the talk, to be padded out with other material from the texts or examples that the student finds interesting, relevant, enlightening, useful, etc.

If you have any questions please feel free to contact me at the email address above.

NB

About the absence of 1-categories before Talk VI: This seminar will complement the course "Categories and Homotopy Theory" 19234201, which starts on the 8th of June. As such, we avoid as much as possible using 1-categories in the classical sense in the first half of the seminar. Moreover, model categories, simplicial categories, homotopy limits, etc only appear at the end of the course, after they have appeared in "Categories and Homotopy Theory". This means that we lose a very important point of view on the subject, but on the other hand, it highlights homotopy theoretic tones of the material.

(∗) About the references: Sometimes I insist on mentioning aspects of the theory which, as stated in the references, go beyond the scope of this course. When I do this, the reference is marked with a star. This is a warning that the language of the literature is not what we are using, or that there is a lot of background material needed to understand the statement as written that we are not going to cover.

Overview and schedule

I. 24.04. Introduction

Topology studies those aspects of spaces which are preserved by stretching and bending, but not tearing or gluing. From this point of view, the surface of a doughnut (with hole) is the same as the surface of a coffee cup (with handle), but these are both different from the surface of a ball. Similarly, the shape of the written number 1 is the same as the shape of the written numbers 3, 5, and 7 but different from 6, 0, and 9, and these are different from 8. The shape of 2 and 4 either fall into the first or second groups depending on how you write them.

A basic tool used to show that two spaces are different, is the fundamental groupoid $\pi_{\leq 1}(X)$ of a space X. This is the set of ways we can move from one point to another of our space leaving a trail of string. Two paths are considered the same if we can slide, stretch, or contract the string path of one to the other without leaving the space, and without moving the start and end points. It is a groupoid because given two paths, one starting where the other finishes, we get a third by concatenating them. The groupoid $\pi_{\leq 1}(X)$ is not changed under deformation, so if two spaces have different $\pi_{\leq 1}$'s, we can conclude that one cannot be obtained from the other by deformation. For example, a circle is different from a sphere, because every path on a sphere can be contracted to a point.

The fundamental groupoid only contains information about holes of "dimension ≤ 1". It can tell that a figure 8 is different from a figure 0, but not that a sphere is different from a point. To do this, we should also use paths of fabric between two string paths. But then we only get "dimension 2" holes, so we should use blocks, etc, etc.

This is a basic example of an ∞-category: The set of continuous maps $\square^n_{top} \to X$, where $0 \leq n < \infty$ and $\square^n_{top} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1\}$, together with

the information of which maps $\square_{top}^{n-1} \to X$ are the face of a map $\square_{top}^{n} \to X$, and which maps $\square_{top}^{n} \to X$ are obtained from a map $\square_{top}^{n-1} \to X$ by just not moving in one direction.

The "∞" refers to the fact that we are allowed any $n < \infty$, and the "category" from the fact that we can concatenate two maps $\square_{top}^{n} \rightrightarrows X$ if an ending face of one agrees with a starting face of the other.

II. 08.05. Simplicial sets (Danijela)

In practice, it is often more practical to work with triangles rather than squares. A *simplicial set* is an abstract combinatorial object, which mimics the ∞-category of a topological space: we have a set K_n for every $0 \leq n < \infty$, which we can think of as maps from an n-dimensional triangle into a space, and various morphisms $\delta_i : K_n \to K_{n-1}, \sigma_i : K_{n-1} \to K_n$ telling us how the triangles fit together.

In this lecture the basic definitions are given, together with some basic examples.

This lecture will cover the following:

See [Danijela's notes (scanned pdf), used with permission].

Another useful reference is: arxiv:0809.4221.

III. 15.05. ∞-Categories (Kristian)

Not only do we get a simplicial set from every topological space, but by using the simplicial set as a recipe to glue various Δ_{top}^{n} together we can get a topological space from every simplicial set. The first part of this lecture describes how using this we can transport the notion of when two topological spaces are "the same" (from the point of view of topology) to define when two simplicial sets are "the same".

Not every simplicial set has the "composition" property possessed by simplicial sets of topological spaces. The second part of this lecture formalises what it means to be able to "compose" simplicies, and defines ∞-categories. It finishes with the observation that the simplicial set of morphisms between an ∞-category is again an ∞-category.

This lecture will cover the following:

See [Lecture notes (pdf)].

IV. 22.05. Joins and slice ∞-categories (Arne)

The *join* of two topological spaces is the topological space we get by adding a path from every point in one to every point in the other. For example, if one space is a circle, and the other a point, we get the shape of an icecream cone (without the icecream). This lecture shows how to define this for simplicial sets. In the

case one space is a single point, the join is called the *cone* for obvious reasons. Joins are needed for the definition of *slice* categories, which are needed for the definition of *limits*, in the next lecture. The "slice" of a morphism of topological spaces $f : X \to Y$ is something like the space of pairs (x, γ) where $x \in X$ is a point and $\gamma : [0, 1] \to Y$ is a path starting from $f(x)$.

This lecture will cover the following:

Define the right and left cone of a directed graph and a partially ordered set.[1] Define the cone of a topological space, and draw the picture [Hat, pp.8-9]. Define the join of two topological spaces, and draw the picture [Hat, p.9]. Define the join of two simplicial sets [Gro, Def.2.11], [HTT, Def.1.2.8.1]. Show that there are isomorphisms $\Delta^i \star \Delta^j \cong \Delta^{i+j+1}$. Define the right cone and left cone of a simplicial set, and describe them explicitly [Gro Exa.2.14], [HTT, Not.1.2.8.4]. Show that the ∞-categories of the cones of a directed graph and partially ordered set are the ∞-categories of their cones. *Prove* that for any two ∞-categories S, S', the join $S \star S'$ is an ∞-category [HTT, Prop.1.2.8.3].

Define the overcategory $C_{/p}$ of a map p by its universal property [Gro, Prop.2.17], [HTT, Prop.1.2.9.2]. Define $C_{/p}$ explicitly [HTT, Proof of Prop.1.2.9.2]. State (without proof) that $C_{/p}$ is an ∞-category [HTT, Prop.1.2.9.3]. Define the undercategory by a universal property, and explicitly [HTT, Rem.1.2.9.5]. Given a morphism of topological spaces $p : Y \to X$, explicitly describe the ∞-category $Sing_\bullet(X)_{/Sing(p)}$. Do the same for directed graphs and partially ordered sets if there is time.

V. 29.05. Limits and colimits in ∞-categories (Robert)

Colimits are a vast generalisation and unification of unions and quotients. They are a way of gluing spaces together. The colimit of a collection of morphisms of ∞-categories is in a precise sense the "supremum" of this collection. Dually, limits are a vast generalisation and unification of intersections, fixed points, and kernels. The limit of a collection of morphisms of ∞-categories is in a precise sense the "infimum" of this collection. (Co)Limits are as basic to category theory as convergence is to analysis, but we will most immediately use them to define *stable ∞-categories*. If we do the Seifert-van Kampen Theorem they will appear there too.

This lecture will cover the following:

Define the ∞-category of topological spaces.[2] Recall the definition of weak equivalence and Kan fibration from Talk III. Claim that a morphism of simplicial sets is a Kan fibration and a weak equivalence if and only if it is a trivial fibration (=trivial Kan fibration) in the sense of [HTT, Exa.2.0.0.2]. Recall that a morphism which is both a weak equivalence and a Kan fibration is called an *acyclic fibration* or *trivial fibration*. Define initial and final objects [Gro, §2.4], [HTT, §1.2.12.3]. Show that the ∞-category of a partially ordered set has an initial (resp. final) object if and only if it has a minimal (resp. maximal) element. (Optional) State the equivalent conditions of [Gro, Prop.2.23] (use the right mapping space [Gro, Rem.16(ii)], [HTT, p.27] as in [HTT, Prop.1.2.12.4]). Show that the one point topological space

[1] For a graph, add one extra vertex and one edge to / from it for every old vertex. For a partially ordered set, add a new element defined to be less than / greater than every old element.

[2] See Example XI.6 at the end of this document.

is a final object in the ∞-category of topological spaces. State that a topological space homotopy equivalent to a one point topological space is a final object. (Optional) Prove this. (Optional) Find sufficient and necessary conditions for a topological space to be a final object.

Define colimits and limits [HTT, Def.1.2.13.4]. Observe that initial (resp. final) objects are colimits (resp. limits) of the empty diagram. Show that limits / colimits in 0-categories (i.e., nerves of partially ordered sets) are infimums / supremums. Define pushout and pullback squares [Gro, Def.2.29].

Define the *homotopy pushout* of a diagram $Z \xleftarrow{f} X \xrightarrow{g} Y$ of topological spaces as $(Y \amalg ([0,1]\times X) \amalg Z)$ modulo the relations $f(X) \sim \{0\}\times X$ and $g(X) \sim \{1\}\times X$. Claim that this gives a pushout square in the ∞-category of topological spaces. (Optional) Show this claim. Define the *homotopy pullback* of a diagram $Z \xrightarrow{f} X \xleftarrow{g} Y$ of topological spaces as $\{(z,\gamma,y) \in Z \times \hom([0,1],X) \times Y : \gamma(0) = f(z), \gamma(q) = g(y)\}$. Claim that this gives a pullback square in the ∞-category of topological spaces. (Optional) Show this claim. (Optional) Do any / all of the following examples in the ∞-category of topological spaces: (co)products, (co)equalisers, mapping (co)telescope.

(Optional) Define cofinal morphisms [HTT, Def.4.1.1.1]. (Optional) State the equivalent conditions of [HTT, Prop.4.1.1.8]. (Optional) Define left Kan extensions along full subcategories [HTT, Def.4.3.2.2]. (Optional) State the existence of left Kan extensions along full subcategories when the target is cocomplete [HTT, Cor.4.3.2.14]. (Optional) Explain why Kan extensions are more complicated along morphisms which are not full inclusions [HTT, §4.3.3]. (Optional) Define left extensions [HTT, Def.4.3.3.1]. (Optional) Define left Kan extensions in general [HTT, Def.4.3.3.2]. (Optional) State that the two definitions are compatible [HTT, Prop.4.3.3.5]. (Optional) Explain how colimits are examples of left Kan extensions. (Optional) Explain how inverse image of (classical) sheaves along a morphism of topological spaces is an example of a left Kan extensions.

Reminder. 08.06. "Categories and Homotopy Theory" 19234201 starts.

VI. 12.06. Monoidal ∞-categories (Karl)

Some of the most interesting topological spaces come equipped with a multiplication, e.g., $GL_n(\mathbb{C})$. This defines a "multiplication" on its associated simplicial set. Monoidal categories are those equipped with a "multiplication". An important more general case is the space of loops of a topological space, written ΩX. Here, the composition is not so straightforward.

Recall that when defining the composition of two paths $\gamma, \gamma' : [0,1] \to X$ in a topological space with $\gamma(1) = \gamma'(0)$, we get the composition $\gamma' \circ \gamma : [0,1] \to X$ by travelling faster along the first map, then faster along the second map, see [HA, Beginning of Chap.5]. So there is a space of composition choices, and any choice is deformable into any other. Recall also that it is important to have this choice, because it is the only way to have associativity $(\gamma'' \circ \gamma') \circ \gamma = \gamma'' \circ (\gamma' \circ \gamma)$. So we want to keep track of the ∞-category of composition choices of n-paths for each $n = 0, 1, \ldots$, as well as the many various functors sending a composition choice of

n paths to a composition choice of k paths, for $n, k = 0, 1, \ldots$. In practice it turns out to be a good idea to use *coCartesian fibrations* to organise this data.

Given a morphism $f : S \to \text{Subsets}(Y)$ from a set S to the set of subsets of a set Y, we can define $X = \{(s, y) : s \in S, y \in f(s)\}$ and instead work with the morphism of sets $p : X \to S; (s, y) \mapsto s$. We can recover f as p^{-1}, but the organisation of the data is a little cleaner. This is the idea behind coCartesian fibrations. Instead of a functor $S \to \infty\text{-}\mathsf{Cat}$ from an ∞-category to the ∞-category of ∞-categories, which in general may be complicated to define and work with, we instead work with a special class of morphisms of ∞-categories called coCartesian fibrations. These are precisely those morphisms which are obtained analogously to the way we got $p : X \to S$ from $f : S \to \text{Subsets}(Y)$.

From this point of view, the ∞-category associated to ΩX is not the assignment of an ∞-category to each n, and a functor to each $[n] \to [k]$, but rather, a coCartesian fibration $C \to \Delta^{op}$ towards the ∞-category associated to the collection of the partially ordered sets $[n]$.

This lecture will cover the following:

Define an inner fibration of simplicial sets [Gro, Def.1.37], [HTT, Def.2.0.0.3]. Define coCartesian morphisms [Gro, Def.4.12], [HTT, Def.2.4.1.1]. Unwrap this definition (i.e., state it in terms of lifting diagrams of inclusions of horns and boundaries of simplicies) [HTT, Rem.2.4.1.4]. Define coCartesian fibrations [Gro, Def. 4.13], [HTT, Def.2.4.2.1].

State the following: "Principle. The ∞-category of coCartesian fibrations towards an ∞-category S is equivalent to the ∞-category of functors from S to the ∞-category of ∞-categories" [HTT, Thm.3.2.0.1 and preceding paragraph]*.

More concretely: Observe that if A is an ∞-category, then $A \to \Delta^0$ is a coCartesian fibration. Show that given two ∞-categories A^0, A^1 and a morphism $A^1 \to A^0$, one can associate a coCartesian fibration $p : N_A(\Delta^1) \to \Delta^1$ such that $p^{-1}(0) = A^0$ and $p^{-1}(1) = A^1$ [HTT, Def.3.2.5.2].

Show that given a coCartesian fibration $p : X \to S$, for every 0-simplex $s \in S_0$, the fibre $p^{-1}(s)$ is an ∞-category. Claim that for any coCartesian fibration $p : X \to \Delta^1$ there exists a morphism $p^{-1}(1) \to p^{-1}(0)$ [HTT, Lem.2.1.1.4], [HTT, Beginning of §2.4]. Show that if $p : X \to S$ is a coCartesian fibration and $\sigma : \Delta^1 \to S$ a morphism, then $p^{-1}(\sigma) \to \Delta^1$ is a coCartesian fibration. Deduce that for any coCartesian fibration $p : X \to S$ and any edge $\sigma : \Delta^1 \to S$, there exists a morphism $p^{-1}(\sigma(1)) \to p^{-1}(\sigma(0))$.

Define the ∞-category[3] $N\Delta^{op}$. Define monoidal ∞-categories [Gro, Def.4.14 and p.48], [DAGII, Def.1.1.2].

Present the following example [Gro, Exa.4.7]*, [DAGII, Def.1.1.1]*: Let G be a group. The set of q-simplicies $(G^{\otimes})_q$ is the set of tuples

$$([m_0] \overset{\alpha_1}{\leftarrow} \ldots \overset{\alpha_q}{\leftarrow} [m_q], ((g_{0,1}, g_{0,2}, \ldots, g_{0,m_0}), \ldots, (g_{q,1}, g_{q,2}, \ldots, g_{q,m_q})))$$

such that for each $0 \leq i < q$ and $0 \leq j < m_i$ we have

[3]See Example XI.4 at the end of this document.

$$g_{i+1,j+1} = g_{i,\alpha_i(j)+1} g_{i,\alpha_i(j)+2} \cdots g_{i,\alpha_i(j+1)}$$

where $[m_0] \leftarrow \ldots \leftarrow [m_q]$ is a q-tuple of $N\Delta^{op}$, and $(g_{i,1}, g_{i,2}, \ldots, g_{i,m_i}) \in G^{m_i}$ for each $i = 0, \ldots, q$. If $\alpha_i(j) = \alpha_i(j+1)$ we interpret the right hand side as the identity element $e \in G$. Note that considering $g_i = (g_{i,1}, \ldots, g_{i,m_i})$ as an element of BG_{m_i}, the demanded equality can be more succinctly written as $g_{i+1} = \alpha^* g_i$. The functor $G^{\otimes} \to N\Delta^{op}$ is projection to the first component. Observe that the preimage of $[1]$ is the discrete ∞-category $G^{\otimes}_{[1]} = G$. Observe that the preimage of the map $\delta_1 : [1] \to [2]; 0, 1 \mapsto 0, 2$ defines the multiplication on G. Observe that the preimage of the map $[1] \to [0]$ determines the identity element of G. Observe that the associativity condition $(xy)z = x(yz)$ of G is implied by the fact that the two compositions $\{0, 3\} \to \{0, 1, 3\} \to \{0, 1, 2, 3\}$ and $\{0, 3\} \to \{0, 2, 3\} \to \{0, 1, 2, 3\}$ are equal. Observe that the associativity condition $((wx)y)z = (wx)(yz) = w(x(y(z))) = w((xy)z) = (w(xy))z$ is also encoded in a similar way. Observe that for each n, the morphism $G^{\otimes}_{[n]} \to (G^{\otimes}_{[1]})^n$ induced by the maps $[1] \to [n]; 0, 1 \mapsto i-1, i$ is an isomorphism.

Present the following example. Let (X, x) be a pointed topological space, and we will define an ∞-category $\Omega X^{\otimes}$. The q-simplicies are tuples

$$([m_0] \overset{\alpha_1}{\leftarrow} \ldots \overset{\alpha_q}{\leftarrow} [m_q], (h_I : \Delta^j_{top} \times \Delta^{m_{i_j}}_{top} \to X)_{I=(i_0 \leq \ldots \leq i_j) \subseteq [q]})$$

where the tuples are required to satisfy:

(1) $h_I(\Delta^j_{top} \times \{e_k\}) = x$ for each corner $\{e_k\} = (0, \ldots, 0, 1, 0, \ldots, 0) \in \Delta^{m_{i_j}}_{top}$ (the 1 is in the kth coordinate).
(2) For every inclusion $I' = (i'_0 \leq \ldots \leq i'_{j'}) \subseteq I = (i_0 \leq \ldots \leq i_j) \subseteq [q]$ the composition $h_I \circ (\Delta^{j'}_{top} \times \Delta^{m_{i'_{j'}}}_{top} \to \Delta^j_{top} \times \Delta^{m_{i_j}}_{top})$ is equal to $h_{I'}$, where the $\Delta^{j'}_{top} \to \Delta^j_{top}$ is induced by the inclusion $I' \subseteq I$ and $\Delta^{m_{i'_{j'}}}_{top} \to \Delta^{m_{i_j}}_{top}$ is induced by $\alpha_{i'_{j'}} \alpha_{i'_{j'}+1} \alpha_{i'_{j'}+2} \ldots \alpha_{i_j} : [m_{i_j}] \to [m_{i'_{j'}}]$.

Observe that the fibre $\Omega X^{\otimes}_{[1]}$ over $[1]$ is the singular simplicial set of the loop space $Sing_\bullet \Omega X$. More generally, observe that the fibre over $[n]$ is the singular simplicial set of the subspace of $\underline{\hom}(\Delta^n_{top}, X)$ consisting of those maps sending the corners to x. Observe that the morphism $\Omega X^{\otimes}_{[n]} \to (\Omega X^{\otimes}_{[1]})^n$ induced by the maps $[1] \to [n]; 0, 1 \mapsto i-1, i$ is a homotopy equivalence but *not* an isomorphism.

Define monoidal and lax monoidal functors [DAGII, Def.1.1.8]. Define algebras objects [DAGII, Def.1.1.14]. Define left-tensored ∞-categories [DAGII, Def.2.1.1]. Give example [DAGII, Exa.2.1.3]. Define module objects [DAGII, Def.2.1.4].

VII. 19.06. Stable ∞-categories. (Vincent)

Two more canonical examples of ∞-categories are the collection of all pointed topological spaces, and the collection of all complexes of vector spaces. The smash product of a pointed topological space with the circle S^1 (called suspension) corresponds to shifting a complex of vector spaces. However, while every complex of vector spaces is the shift of some other complex, not every pointed topological space is the suspension of some other pointed topological space. Using colimits,

there is a notion of smash-product-with-S^1 in any (pointed) ∞-category, and the *stable* ∞-categories are those in which this procedure is an equivalence.

This lecture will cover the following:

Define the ∞-category Vec of bounded complexes of vector spaces.[4] Define the ∞-category Top_* of pointed topological spaces. Define pointed ∞-categories [Gro, Def.5.1], [DAGI, Def.2.1]. Observe that the zero vector space is both an initial and final object of Vec. Observe that in fact, any complex homotopy equivalent to zero is both an initial and final object of Vec. Observe that the point is both an initial and final object of Top_*. Recall that Top is not a pointed ∞-category. Define triangles in pointed ∞-categories [Gro, p.60], [DAGI, Def.2.4]. Define exact and coexact triangules in pointed ∞-categories [Gro, Def.5.5], [DAGI, Def.2.4]. Observe that triangles in Vec with corners $K_\bullet \xrightarrow{f} K'_\bullet \to Cone(f)$ are both exact and coexact. In particular, if $0 \to V \to V' \to V'' \to 0$ is an exact sequence of vector spaces, then $V \to V' \to V''$ is an exact and coexact triangle. Observe that the triangles $X \xrightarrow{f} Y \to Cone(f)$ in Top_* are coexact. Observe that the triangles $Fib(f) \to X \xrightarrow{f} Y$ in Top_* are exact. Give an example of a coexact triangle in Top_* which is not exact, and an exact triangle which is not coexact. Define kernel and cokernels in pointed ∞-categories [DAGI, Def.2.6]. Define stable ∞-infinity categories [Gro. Def.5.11], [DAGI, Def.2.9]. Observe that Vec is stable, but Top_* is not stable.

Define the homotopy category $h(C)$ of an ∞-category C using the construction $\pi(C)$ of [HTT], (resp. $\tau_1(C)$ of [Gro]), [Gro, Prop.1.15]*, [HHT, §1.2.3]*. Observe that morphisms of $h(Vec)$ are morphisms of chain complexes up to chain homotopy, and the morphisms of $h(Top)$ are the morphisms of topological spaces up to homotopy. Observe that $h(C)$ is a 1-category.

Define the suspension Σ and loop Ω functors of a pointed ∞-category admitting pushouts and pullbacks [Gro, Def.5.8 and preceding text], [DAGI, Rem.3.2]. Explain that in such an ∞-category C, for any objects $X, Y \in C_0$ the set of morphisms $\hom_{hC}(\Sigma X, Y)$ in the homotopy category is a group, and the set of morphisms $\hom_{hC}(\Sigma\Sigma X, Y)$ is an abelian group.[5] State that if C is a stable ∞-category, then hC, equipped with the class of (co)exact triangles, satisfies the axioms (TR1)-(TR4) [Gro, Thm.5.15], [DAGI, Def.3.1, Thm.3.11]. Prove any or all of these axioms (note the proof of the octahedral axiom (TR4) is surprisingly simple in this setting).

VIII. 26.06. ∞-Topoi (Georg)

Classical (1-)topoi arise in a number of different ways:

[4] See Example XI.7 at the end of this document.

[5] Step 1. Observe that ΣX is the colimit of any diagram of the form $0 \leftarrow X \rightarrow X \leftarrow X \rightarrow \ldots \leftarrow X \rightarrow 0$, and $\Sigma X \sqcup \Sigma X \sqcup \cdots \sqcup \Sigma X$ is the colimit of any diagram of the form $0 \leftarrow X \rightarrow 0 \leftarrow X \rightarrow \ldots \leftarrow X \rightarrow 0$.
Step 2. Observe that by sending some X's to zero in such a diagram, we obtain a map $\Sigma X \to \Sigma X \sqcup \Sigma X \sqcup \cdots \sqcup \Sigma X$.
Step 3. Observe that this induces maps $\hom_{hC}(\Sigma X, Y) \times \cdots \times \hom_{hC}(\Sigma X, Y) \to \hom_{hC}(\Sigma X, Y)$.
Step 4. Observe that any zero morphism $\Sigma X \to 0 \to Y$ induces a unit element for this operation.
Step 5. Observe that $\hom_{hC}(\Sigma\Sigma X, Y)$ admits two multiplications which are compatible. Therefore, by the Eckman-Hilton Lemma they are the same. Consequently, this multiplication is commutative.

(1) As generalisations of topological spaces.
(2) As generalisations of set theory as a logical foundation of mathematics.
(3) As a class of cocomplete (1-)categories that can be defined using generators and relations.

Recall that not every ∞-category is cocomplete, that is, not every ∞-category admits all colimits. A canonical and practically useful way to embed an ∞-category C in a cocomplete ∞-category is to consider the ∞-category $P(C) = \mathrm{Fun}(C^{op}, \mathcal{S})$ of functors from C^{op} to the ∞-category of spaces $\mathcal{S}$. Objects of this category are called presheaves for historical reasons.

In the case that C is the nerve $NOp(X)$ of the partially ordered set $Op(X)$ of open sets of a topological space X, the category $P(NOp(X))$ contains a very interesting full subcategory—the category $Shv(NOp(X))$ of *sheaves*. A functor $F : NOp(X)^{op} \to \mathcal{S}$ is a sheaf if for any open covering $U = \{U_i \subseteq V\}_{i \in I}$ of some open $V \subseteq X$ which is closed under intersection (i.e., for all $U_i, U_j \in U$ we have $U_i \cap U_j \in U$), we can reconstruct $F(V)$ as the limit $\lim F|_{NU}$ of the induced diagram $NU^{op} \to \mathcal{S}$. Every presheaf can be forced to be a sheaf—there is a functor $P(NOp(X)) \to Shv(NOp(X))$. In fact, this is the universal functor which forces two presheaves to be equivalent if for every point $x \in X$ they agree on a sufficiently small neighbourhood.

If we think of $P(NOp(X))$ as the cocomplete ∞-category generated by $NOp(X)$, then we can think of the functor $P(NOp(X)) \to Shv(NOp(X))$ as enforcing certain relations, in this case, the relation that two presheaves "agreeing" on an open cover are in fact equivalent.

∞-Categories built in this way—cocompletion of some smaller ∞-category and then forcing relations of a certain kind—arise in a much more general setting than classical topological spaces.

This lecture will be a more advanced talk, covering material of the speaker's choice.

IX. 03.07. ∞-Operads (André)

We have already mentioned that not every topological space is of the form ΩX for some topological space X. For example, each ΩX is connected. But not every connected topological space is of the form ΩX either. In the homotopy category, there is a canonical group structure on ΩX, but the 7-sphere S^7 admits such a structure in the homotopy category (induced by the octonians) but cannot be a loop space by a Steenrod algebra operations argument. To recognise that ΩX is a loop space, we recall that for each n, there is a contractible space of "multiplications" $(\Omega X)^n \to \Omega X$ satisfying various conditions. Such an object is called a group like monoid, over the $\mathbb{E}_1$-operad (the operad is the collection of contractible spaces of multiplications).

In this talk, we work towards understanding the statement of this recognition principle—that the functor sending a pointed space to its associated grouplike $\mathbb{E}_1$-operad is an equivalence of categories.

This lecture will cover the following: TBA

X. 10.07. Spectra (Alex)

We studied stable ∞-categories in Talk VII and observed that not every ∞-category is stable. There is however, a universal stable ∞-category associated to any ∞-category. More precisely, for any (presentable) ∞-category C, there is a colimit preserving functor $C \to \mathcal{S}(C)$ such that the ∞-category $\mathrm{Fun}^L(C, D)$ of colimit preserving functors from C to any (presentable) stable ∞-category D is equivalent to $\mathrm{Fun}^L(\mathcal{S}(C), D)$, the category of colimit preserving functors out of $\mathcal{S}(C)$.

There are at least two ways to do this. One way is as follows: we would like the loop functor $\Omega : C \to C$ to be an equivalence. So one could take the inverse limit of the diagram $\cdots \stackrel{\Omega}{\to} C \stackrel{\Omega}{\to} C \stackrel{\Omega}{\to} C$ in the ∞-category of ∞-categories. There is a more concrete description of this ∞-category via *spectra*. A spectrum, or rather, an Ω-spectrum in *Top* is, classically, a sequence of topological spaces $(X_0, X_1, \dots)$ with homotopy equivalences $X_i \to \Omega X_{i+1}$. If there is time we may discuss the ∞-category version of this description. One can think of X_i as the "ith component" in the limit described above. It is informative to observe that if we replace a complex of vector spaces K by its shifted truncations $(\tau_{\leq i}K)[i]$ and canonical morphisms $(\tau_{\leq i}K)[-i] \to (\tau_{\leq i+1}K)[-i-1]$ we don't lose any information. Indeed, the ∞-category of chain complexes is equivalent to the ∞-category of Ω-spectra of bounded-below-zero chain complexes.

Another way of constructing the stabilisation of an ∞-category is via *excisive functors* from $\mathcal{S}_*^{fin}$, the category of pointed objects in the smallest sub-∞-category of spaces $\mathcal{S}$ containing the terminal object. An excisive functor is one sending pushout squares to pullback squares. Given such a functor F, we obtain a Ω-spectrum in the classical sense by defining $X_n = F(\Sigma^n S^0)$, and the structural morphisms come from the property $F(\Sigma^n S^0) = \Omega F(\Sigma^{n-1} S^0)$. Conversely, given any Ω-spectrum one can extend this to a functor on all of $\mathcal{S}_*^{fin}$.

This latter procedure, using excisive functors, is where this talk will start. This is also useful foundation work for learning about Goodwillie calculus. If there is time, we will discuss the Ω-spectrum point of view and observe that the two constructions produce equivalent categories.

This lecture will cover the following: TBA

XI. 17.07. Simplicial model categories (Tommaso)

Simplicial categories and model categories are older notions that predate ∞-categories. What we have been calling ∞-categories are more properly called *quasi-categories*, and they are just one of many equivalent theories of ∞-categories, simplicial categories being another theory.

A simplicial category is like a 1-category, but instead of a set of morphisms, we have a simplicial set of morphisms associated to every pair of objects. There is an equivalence between the theory of quasi-categories and the theory of simplicial categories in a very precise sense. This equivalence is very useful. There are many constructions, such as the Yoneda embedding from Talk VIII, which are trivial in the category of simplicial categories. Another example is the construction of (co)limits. While (co)limits in ∞-categories are defined by a universal property

rather than a construction, in simplicial categories there is a a concrete construction. Moving to the world of simplicial categories gives a way to calculate (co)limits in ∞-categories.

While there is a quasi-category associated to any model category, there may be many model categories associated to a given (presentable) quasi-category. Choosing a model category giving a quasi-category is analogous to choosing a basis for a vector space. Model categories also give a language for comparing various theories—there is a model category of quasi-categories, and a model category of simplicial categories, and a Quillen equivalence between these model categories.

This lecture will cover the following: TBA

Examples of ∞-categories

Example XI.1. Recall that a partially ordered set is a set P_0 together with a antisymmetric, transitive, reflexive binary relation $P_1 \subseteq P_0 \times P_0$. The set NP_n of n-simplicies of the *nerve* NP of a partially ordered set is

$$NP_n = \{(x_0, \dots, x_n) \in P_0^{n+1} = P_0 \times \cdots \times P_0 : (x_i, x_{i+1}) \in P_1 \text{ for } 0 \leq i < n\}.$$

In other words, ordered tuples $x_0 \leq \cdots \leq x_n$. The face morphisms are

$$d_i : (x_0, \dots, x_n) \mapsto (x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

and the degeneracy morphisms are

$$s_i : (x_0, \dots, x_n) \mapsto (x_0, \dots, x_i, x_i, \dots, x_n).$$

Note that we could also have written NP_n as the iterated fibre product $P_1^{\times_{P_0} n} = P_1 \times_{P_0} \cdots \times_{P_0} P_1$.

Example XI.2. Recall that a *group* is a set G_1 equipped with a multiplication $G_1 \times G_1 \to G_1$ which is associative, and every element $g \in G_1$ admits an inverse. The *classifying space* BG of G is the simplicial set whose set of n-simplicies is

$$G_n = G_1^n = G_1 \times \cdots \times G_1.$$

The set G_0 has a single element. The face morphisms are

$$d_i : (g_1, \dots, g_n) \mapsto \begin{cases} (g_2, \dots, g_n) & i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & 0 < i < n \\ (g_1, \dots, g_{n-1}) & i = n \end{cases}$$

and the degeneracy morphisms are

$$s_i : (g_1, \dots, g_n) \mapsto (g_1, \dots, g_i, e, g_{i+1}, \dots, g_n).$$

The nerve of a partially ordered set and the classifying space of a group are special cases of the nerve of a (small classical) category.

Example XI.3. Recall that a (small classical) category is a set C_0 whose elements are called *objects*, a set C_1 whose elements are called *morphisms*, four set maps

$$\mathrm{id} : C_0 \to C_1, \qquad s, t : C_1 \rightrightarrows C_0,$$

$$\circ : \{(f, g) \in C_1 \times C_1 : s(g) = t(f)\} \to C_1; \qquad (f, g) \mapsto g \circ f$$

called *identity*, *source*, *target*, and *composition* respectively, such that the multiplication is associative (so $h \circ (g \circ f) = (h \circ g) \circ f$ when $s(g) = t(f), s(h) = t(g)$), for every $x \in C_0$ we have $s(e_x) = t(e_x)$, and the images of C_0 are identity elements (so

$\mathrm{id}_{t(f)} \circ f = f$ and $f \circ \mathrm{id}_{s(f)} = f$). The *nerve* NC of a category is the simplicial set whose n-simplicies for $n \geq 1$ are

$$NC_n = \{(f_1, \ldots, f_n) \in C_1^n : s(f_{i+1}) = t(f_i)\}$$

and set of 0-simplices is C_0. The face morphisms are

$$d_0 = t, d_1 = s : C_1 \to C_0 \text{ for } n = 1,$$

$$d_i : (f_1, \ldots, f_n) \mapsto \begin{cases} (f_2, \ldots, f_n) & i = 0 \\ (f_1, \ldots, f_{i+1} \circ f_i, \ldots, f_n) & 0 < i < n \\ (f_1, \ldots, f_{n-1}) & i = n \end{cases}$$

and the degeneracy morphisms are

$$s_i : (f_1, \ldots, f_n) \mapsto (f_1, \ldots, f_{i-1}, \mathrm{id}_{t(f_{i-1})} = \mathrm{id}_{s(f_i)}, f_i, \ldots, f_n).$$

Here is a special case of the nerve of a category.

Example XI.4. The q-simplices of Δ^{op} are sequences $[m_0] \leftarrow \ldots \leftarrow [m_q]$ of non-decreasing morphisms where $[m] = \{0 \leq 1 \leq \cdots \leq m\}$ for $m = 0, 1, 2, \ldots$. Boundaries are given by removing an $[m_i]$ and composing morphisms, degeneracies are given by inserting an identity morphism.

Example XI.5. The 2-category Cat of (small classical) categories is the following simplicial set. An n-simplex is:

(1) A tuple $(C_0, \ldots, C_n)$ of $n+1$ (small classical) categories (cf. Example XI.3).
(2) A tuple $(F_{ij} : C_i \to C_j)_{0 \leq i < j \leq n}$ of functors.
(3) A tuple $(\eta_{ijk} : F_{ik} \Rightarrow F_{jk} \circ F_{ij})_{0 \leq i < j < k \leq n}$ of natural transformations.
(4) The natural transformations are required to satisfy the compatibility condition: for every $0 \leq i, j, k, \ell \leq n$ the square

$$\begin{array}{ccc} F_{i\ell} & \xrightarrow{\eta_{ij\ell}} & F_{j\ell}F_{ij} \\ {\scriptstyle \eta_{ik\ell}}\downarrow & & \downarrow{\scriptstyle \eta_{\eta_{jk\ell}F_{ij}}} \\ F_{k\ell}F_{ik} & \xrightarrow[F_{k\ell}\eta_{ijk}]{} & F_{k\ell}F_{jk}F_{ij} \end{array}$$

of natural transformations commutes.

Example XI.6. The ∞-category Top of topological spaces is the following simplicial set. An n-simplex is:

(1) A tuple $(X_0, \ldots, X_n)$ of $n+1$ topological spaces.
(2) An tuple of morphisms

$$(h_{i,j} : X_i \times \square_{top}^{j-i-1} \to X_j)_{0 \leq i < j \leq n}$$

where $\square_{top}^m = \{(t_1, \ldots, t_m) \in \mathbb{R}^m : 0 \leq t_i \leq 1\}$.
(3) The morphisms $h_{i,j}$ are required to satisfy the compatibility condition: For every $0 \leq i < j < k \leq n$, we should have

$$\begin{aligned} &h_{i,k}(x, (s_1, \ldots, s_{j-i-1}, 1, t_1, \ldots, t_{k-j-1})) \\ &= h_{j,k}(h_{i,j}(x, (s_1, \ldots, s_{j-i-1})), (t_1, \ldots, t_{k-j-1})) \end{aligned}$$

for all $x \in X_i$, $(s_1, \ldots, s_{j-i-1}) \in \square_{top}^{j-i-1}$, $(t_1, \ldots, t_{k-j-1}) \in \square_{top}^{k-j-1}$.

Notice that a tuple $((X_0,\dots,X_n),(h_{ij})_{0\le i<j\le n})$ defines a morphism

$$f_{ij}(-) \stackrel{def}{=} h_{ij}(-,(0,0,\dots,0)) : X_i \to X_j$$

for each $0 \le i < j \le n$. Moreover, for every $i < i_1 < i_2 < \cdots < i_k < j$ the compatibility conditions imply that

$$h_{ij}(-, e_{i_1} + \cdots + e_{j_k}) = f_{i_k,j} \circ f_{i_{k-1},i_k} \circ \cdots \circ f_{i,i_1} : X_i \to X_j$$

where $e_{i'} = (0,\dots,0,1,0,\dots,0)$ is the i'th standard basis vector of $\mathbb{R}^{j-i}$. So we can interpret h_{ij} as a homotopy between all the possible compositions of the f's with f_{ij} at the "lowest" corner of $\square^{j-i-1}_{top}$ and $f_{j-1,j} \circ f_{j-2,j-1} \circ f_{i+1,i+2} \circ f_{i,i+1}$ at the "highest" corner. The compatibility conditions then can be interpreted as asking that these homotopies are compatible with all compositions.

The face morphisms are

$$d_k : (X_0,\dots,X_n,h_{i,j}) \mapsto (X_0,\dots,X_{k-1},X_{k+1},\dots,X_n,h'_{i,j})$$

where

$$h'_{i,j}(x,t) = \begin{cases} h_{i,j}(x,t) & i<j<k \\ h_{i,j+1}(x,(t_1,\dots,t_{k-i-1},0,t_{k-i},\dots,t_{j-i-1})) & i<k\le j \\ h_{i+1,j+1}(x,t) & k\le i<j. \end{cases}$$

The degeneracy morphisms are

$$d_k : (X_0,\dots,X_n,h_{i,j}) \mapsto (X_0,\dots,X_k,X_k,\dots,X_n,h'_{i,j})$$

where

$$h'_{i,j}(x,t) = \begin{cases} h_{i,j}(x,t) & i<j\le k \\ h_{i,j-1}(x,(t_1,\dots,t_{k-i-1},t_{k-i+1},\dots,t_{j-i-1})) & i\le k<j \\ h_{i-1,j-1}(x,t) & k<i<j. \end{cases}$$

Here, we interpret $h_{i,i}$ as id_{X_i}.

Note that every sequence of continuous homomorphisms $X_0 \stackrel{f_1}{\to} \cdots \stackrel{f_n}{\to} X_n$ defines an n-simplex: choose $h_{i,j}$ to be the composition $X_i \times \square^{j-i-1} \to X_i \stackrel{f_{i+1}}{\to} X_{i+1} \stackrel{f_{i+2}}{\to} \cdots \stackrel{f_j}{\to} X_j$ (i.e., the trivial homotopy).

Example XI.7. Define $\square^n_{vec}$ to be the (homological) complex of vector spaces which in degree q, has one basis vector for every sequence of subsets

$$(\varnothing \subseteq I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_q \subseteq \{1,\dots,n\}).$$

Differentials are alternating sums of the maps induced by forgetting the ith I_i. Another way to think about this is, the basis vectors of $(\square^n_{vec})_q$ are the non-degenerate q-simplicies of $N(\mathrm{Subsets}(\{1,\dots,n\}))$ (which, not coincidentally, is isomorphic to the n-fold cartesian product $(\Delta^1)^n$). The map $(\square^n_{vec})_q \to (\square^n_{vec})_{q-1}$ is the alternating sum of the maps induced by the face morphisms $N(\mathrm{Subsets}(\{1,\dots,n\}))_q \to N(\mathrm{Subsets}(\{1,\dots,n\}))_{q-1}$

A q-simplex of the ∞-category Vec is:

(1) A tuple $(K^{(0)},\dots,K^{(q)})$ of $q+1$ complexes of vector spaces.
(2) A tuple of morphisms

$$(h_{i,j} : K^{(i)} \otimes \square^{j-i-1}_{vec} \to K^{(j)})_{0\le i<j\le q}.$$

(3) The morphisms $h_{i,j}$ are required to satisfy the compatibility condition: For every $a, b \geq 1$, the restriction of $h_{i,i+a+b}$ to $K^{(i)} \otimes \square_{vec}^{a-1} \otimes \square_{vec}^{b-1} \subseteq K^{(i)} \otimes \square_{vec}^{a+b-1}$ is the composition $h_{i+a,i+a+b} \circ (h_{i,i+a} \otimes \mathrm{id}_{\square_{vec}^{b-1}})$. Here, the inclusion $\square_{vec}^{a-1} \otimes \square_{vec}^{b-1} \subseteq \square_{vec}^{a+b-1}$ is induced by the identification of $\{1, \ldots, b-1\}$ with $\{a+1, a+2, \ldots, a+b-1\}$ and the morphism

$$\mathrm{Subsets}(\{1, \ldots, a-1\}) \times \mathrm{Subsets}(\{a+1, \ldots, b+a-1\}) \to \mathrm{Subsets}(\{1, \ldots, a+b-1\})$$
$$I, J \mapsto I \cup \{a\} \cup J$$

As in the case of topological spaces, notice that given a tuple as above, for each $0 \leq i < j \leq n$ we have a morphism $f_{ij} : K^{(i)} \to K^{(j)}$ induced by the inclusion $\square_{vec}^0 \subseteq \square_{vec}^{j-i-1}$ corresponding to the empty set $(\varnothing \subseteq \{1, \ldots, n\})$. The compatibilities enforce that the inclusions $\square_{vec}^0 \subseteq \square_{vec}^{j-i-1}$ must be equal to the various compositions $K^{(i)} \to K^{(i_1)} \to \ldots K^{(j)}$ of the f's.

For example, $h_{i,i+2}$ is a chain homotopy between $f_{i,i+2} : K^{(i)} \to K^{(i+2)}$ and $f_{i+1,i+2} \circ f_{i,i+1} : K^{(i)} \to K^{(i+1)} \to K^{(i+2)}$. The other h's can be interpreted as higher chain homotopies between the chain homotopies.

Example XI.8. Define $\square^n = (\Delta^1)^n$ to be the n-fold cartesian product. Note there is a canonical isomorphism $\square^n \cong N(\mathit{Subsets}(\{1, \ldots, n\})$.

An n-simplex of $\mathcal{S}$, the ∞-category of spaces is:

(1) A tuple $(K^{(0)}, \ldots, K^{(q)})$ of $q+1$ Kan complexes.
(2) A tuple of morphisms
$$(h_{i,j} : K^{(i)} \otimes \square^{j-i-1} \to K^{(j)})_{0 \leq i < j \leq q}.$$
(3) The morphisms $h_{i,j}$ are required to satisfy the compatibility condition: For every $a, b \geq 1$, the restriction of $h_{i,i+a+b}$ to $K^{(i)} \times \square^{a-1} \times \square^{b-1} \subseteq K^{(i)} \otimes \square^{a+b-1}$ is the composition $h_{i+a,i+a+b} \circ (h_{i,i+a} \otimes \mathrm{id}_{\square^{b-1}})$. Here, the inclusion $\square^{a-1} \times \square^{b-1} \subseteq \square^{a+b-1}$ is induced by the identification of $\{1, \ldots, b-1\}$ with $\{a+1, a+2, \ldots, a+b-1\}$ and the morphism

$$\mathrm{Subsets}(\{1, \ldots, a-1\}) \times \mathrm{Subsets}(\{a+1, \ldots, b+a-1\}) \to \mathrm{Subsets}(\{1, \ldots, a+b-1\})$$
$$I, J \mapsto I \cup \{a\} \cup J$$

As in the case of topological spaces, notice that given a tuple as above, for each $0 \leq i < j \leq n$ we have a morphism $f_{ij} : K^{(i)} \to K^{(j)}$ induced by the inclusion $\square^0 \subseteq \square^{j-i-1}$ corresponding to the empty set $(\varnothing \subseteq \{1, \ldots, n\})$. The compatibilities enforce that for each subset $\{i_1, \ldots, i_k\} \subseteq I$, the morphism associated to the inclusion $\Delta^0 \subseteq \square^{j-i-1}$ corresponding to I must be equal to the composition $K^{(i)} \to K^{(i_1)} \to \ldots K^{(j)}$ of the f's.

For example, $h_{i,i+2}$ is a homotopy between $f_{i,i+2} : K^{(i)} \to K^{(i+2)}$ and $f_{i+1,i+2} \circ f_{i,i+1} : K^{(i)} \to K^{(i+1)} \to K^{(i+2)}$. The other h's can be interpreted as higher homotopies between the homotopies.

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