∞ - Operads

∞-analogue of colored operads

<u>Def:</u> A <u>colored operad</u> $\mathcal{O}$ consists of a collection of colors, if $\{X_i\}, Y$ are colors, sets of morphisms (multilinear maps) $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}, Y)$, a composition

$$\prod_{j\in J} \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{I_j}, Y_j) \times \mathrm{Mul}_{\mathcal{O}}(\{Y_k\}_{k\in J}, Z) \longrightarrow \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{\cup I_j}, Z)$$

This composition should be associative and have identities.
The category of colored operads is called COp

<u>Rmk:</u> There is a fully faithful functor $\mathrm{Cat} \hookrightarrow \mathrm{COp}$, by sending a category $\mathcal{C}$ to the operad with colors the objects of $\mathcal{C}$, and $\mathrm{Mul}(\{X\}, Y) = \hom(X,Y)$, $\mathrm{Mul}(\{X_i\}_{i\in I}, Y) = \emptyset$ for $|I| \geq 1$. This functor has a right adjoint with
$\hom(X,Y) := \mathrm{Mul}_{\mathcal{O}}(\{X\}, Y)$.

There is also another functor that turns colored operads into categories:

<u>Construction:</u> If $\mathcal{O}$ is a colored operad, define a category $\mathcal{O}^{\otimes}$ by:

- $\mathrm{ob}(\mathcal{O}^{\otimes}) = \{$ ~~colors~~ finite sequences of colors of $\mathcal{O}\}$
- $\hom(\{X_i\}_{i\in\langle n\rangle^0}, \{Y_j\}_{j\in\langle m\rangle^0}) = \left\{(\alpha, \phi_j) : \begin{array}{l} \alpha: \langle n\rangle \to \langle m\rangle \in \mathrm{Fin}_* \\ \phi_j \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{\alpha^{-1}(j)}, Y_j) \end{array}\right\}$

where $\langle n\rangle = \{*, 1, \ldots, n\}$, $\langle n\rangle^0 = \{1, \ldots, n\}$ and Fin_* is the category of pointed finite sets (i.e. $\alpha(*) = *$)

$\mathcal{O}^{\otimes}$ is equipped with a forgetful functor $p: \mathcal{O}^{\otimes} \to \mathrm{Fin}_*$ that has the property $\mathcal{O}^{\otimes}_{\langle n\rangle} = (\mathcal{O}^{\otimes}_{\langle 1\rangle})^n$

<u>Question:</u> given any category $\mathcal{C}$ with a functor $\pi: \mathcal{C} \to \mathrm{Fin}_*$ and $\mathcal{C}_{\langle n\rangle} \cong (\mathcal{C}_{\langle 1\rangle})^n$, when is $\mathcal{C} \cong \mathcal{O}^{\otimes}$ with

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr] & & \mathcal{O}^{\otimes} \arrow[dl] \\ & \mathrm{Fin}_* & \end{tikzcd}$$

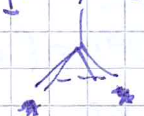
for some c. operad $\mathcal{O}$?

<u>Def</u>: $f: \langle m\rangle \to \langle n\rangle \in \mathcal{F}in_*$ is called <u>inert</u> if every element i of $\langle n\rangle^\circ$ has $f^{-1}(i) = \{g(i)\}$, i.e. if $|f^{-1}(i)| = 1$

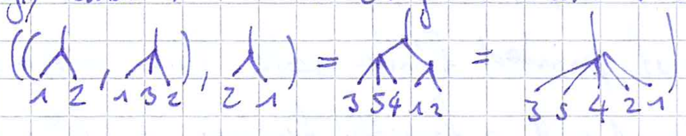
Now, if $C^\otimes$ is a category with a functor $p: C^\otimes \to \mathcal{F}in_*$ s.t. $C^\otimes_{\langle n\rangle} \simeq (C^\otimes_{\langle 1\rangle})^n$, if $f \in \mathcal{F}in_*$ is inert, then f has a coCartesian lift $f_!: C^\otimes_{\langle m\rangle} \to C^\otimes_{\langle n\rangle}$, and morphisms lying over f are uniquely determined by morphisms lying over g^i of ($g^i: \langle n\rangle \to \langle 1\rangle$ is the map sending i to 1 and everything else to $*$), i.e. $\hom^f_{C^\otimes}(C, C') \cong \prod_i \hom^{g^i \circ f}(C, C_i')$, then there exists a unique colored operad $\mathcal{O}$ s.t. $C^\otimes \cong \mathcal{O}^\otimes$.

<u>Def</u> (∞-operad): An ∞-operad is a ∞-functor $p: \mathcal{O}^\otimes \to N(\mathcal{F}in_*)$ of ∞-categories with all the above (translated to the ∞-setting), and $\hom^f_{\mathcal{O}^\otimes}$ is now the connected component of morphisms lying over f, and "$\cong$" is now a homotopy equivalence.

<u>Ex</u>: - If $\mathcal{O}^\otimes$ is any colored operad, $N(\mathcal{O}^\otimes) \to N(\mathcal{F}in_*)$ is an ∞-operad. Let $Comm^\otimes := N(\mathcal{F}in_*)$ be the commutative operad.

- Ass has one color, $*$, and $Mul_{Ass}(\{*\}_I, *) = \{$lin. orderings on $I\}$ or alternatively the set of trees  if $|I| = n$ with n leaves and one internal vertex.

Composition of the trees works by putting leaves and roots together accordingly and then contracting internal vertices accordingly.

(ex: 

The ordering corresponding to a tree is the order of leaves from left to right.

An associative algebra is a map of operads $Ass \to C^\otimes$ to a sym. monoidal cat. (whatever that means), i.e. it induces the usual commutative diagrams.

$Ass^{\otimes} = N(Ass)$ is the corresponding ∞-operad.

- <u>little k-cubes operad $\mathbb{E}_k$</u>: Has as n-simplices tuples

$$(X_0, \dots, X_n, \{h_{ij} : \square_{top}^{j-i-1} \to \coprod_a \coprod_{f_a : X_a \to X_{a+1}} \prod_{e \in X_{a+1}} \mathrm{Rect}(\square_{top}^k \times f^{-1}(e), \square_{top}^k)\})$$

(h_{ij} have appropriate conditions)

$\mathrm{Rect}(\square_{top}^k \times I, \square_{top}^k)$ is the set of rectilinear embeddings, which are continuous maps $\square_{top}^k \times I \to \square_{top}^k$ (where I has the discrete topology) that are ~~linear~~ affine embeddings (for example $\square_{top}^2 \times \{1,2,3\} \to \square_{top}^2$ by )

- We clearly have a forgetful functor $\mathbb{E}_k \to N(Fin_*)$.
- $\mathbb{E}_1^{\otimes} \cong Ass^{\otimes}$, because an n-simplex in $\mathbb{E}_1^{\otimes}$ gives a lin. ordering on n
- There is a functor $\mathbb{E}_k^{\otimes} \times \mathbb{E}_{k'}^{\otimes} \to \mathbb{E}_{k+k'}^{\otimes}$ which induces an iso.

  $\mathbb{E}_k^{\otimes} \otimes \mathbb{E}_{k'}^{\otimes} \to \mathbb{E}_{k+k'}^{\otimes}$ (whatever this means)

<u>Def</u>: $p : \mathcal{O}^{\otimes} \to N(Fin_*)$ ∞-operad, f a morphism in $\mathcal{O}^{\otimes}$. Then f is called inert if $p(f)$ is inert & f is p co Cart.

<u>Def</u>: A map of ∞-operads is a map of simp. sets $\mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ that makes the triangle commute and carries inert morphisms to inert morphisms. $Alg_{\mathcal{O}}(\mathcal{O}')$ is the full subcategory of $Fun_{N(Fin_*)}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$ spanned by morphisms of ∞-operads. $Alg_{\mathcal{O}}(\mathcal{O}')$ is also sometimes called $Mon_{\mathcal{O}}(\mathcal{O}')$

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[r] \arrow[dr] & \mathcal{O}'^{\otimes} \arrow[d] \\ & N(Fin_*) \end{tikzcd}$$

<u>Rmk</u>: If $C^{\otimes}$ is sym. cart. monoidal ∞-cat, $\mathcal{O}^{\otimes}$ ∞-operad, then $Alg_{\mathcal{O}}(C)$ are the algebras you think of.

Now, consider the category of pointed spaces S_*. If X is a space, ΩX is an $\mathbb{E}_k$-monoid, so there is a functor $\Omega^k : S_* \to Mon_{\mathbb{E}_k}(S)$

this is almost an equivalence except:

- Ω^k does not see homotopy groups $< k$
- $\pi_0(\Omega^k X) \cong \pi_k(X)$, so the image of Ω^k has a group structure on π_0 (in general, π_0 is only a monoid)

<u>Solution</u>: Define grouplike monoids: G ass. monoid in $C^\otimes$ with mult. m.
~~m~~ Then G is called grouplike if $(p_1, m), (p_2, m) : G \times G \to G \times G$ are equivalences. $\rightsquigarrow \mathrm{Mon}^{gp}_{Ass}(C)$ of grouplike ass. monoids.

Because $\mathbb{E}_1 \cong Ass$, we define $\mathrm{Mon}^{gp}_{\mathbb{E}_1}(C)$ in the obvious way. Using the embedding $\mathbb{E}_1 \hookrightarrow \mathbb{E}_k$, a $\mathbb{E}_k$ monoid is called grouplike if $\mathbb{E}_1 \hookrightarrow \mathbb{E}_k \to C$ is grouplike. This does not depend on the choice of embedding $\mathbb{E}_1 \hookrightarrow \mathbb{E}_k$.

<u>Main thm</u>: $\Omega^k : S_*^{\geq k} \to \mathrm{Mon}^{gp}_{\mathbb{E}_k}(S)$ is an equiv. of ∞-categories, where $S_*^{\geq k}$ is the ∞-cat. of pointed spaces that have $\pi_i = 1$ for $i < k$.

<u>Rmk</u>: This is still true if S is replaced by any topos and Ω^k by the k-th cobar construction.