The homotopy category

Def Let $X_\bullet$ be a simplicial set. If $x, y \in X_0$,

let $\mathrm{Arr}(x,y) = \{ f : \Delta^1 \to X_\bullet : f(0) = x,\ f(1) = y \}$

We let $id_x \in \mathrm{Arr}(x,x)$ be the 1-simplex

$$\Delta^1 \longrightarrow \Delta^0 \xrightarrow{x} X_\bullet$$

For $f, g \in \mathrm{Arr}(x,y)$ and a 2-simplex $\sigma : \Delta^2 \to X$ s.t.

$$\begin{tikzcd} & y \arrow[d, "id"] \\ x \arrow[ur, "f"] \arrow[r, "g"'] & y \end{tikzcd}$$

we say σ is a homotopy from f to g.

Prop ~~For~~ Homotopy defines an equivalence relation on $\mathrm{Arr}(x,y)$ if $\mathcal{C}$ is an ∞-category

Proof Let ~~$f : x \to y$~~ $f \in \mathrm{Arr}(x,y)$ be an arrow. Then

$$\Delta^2 \xrightarrow{s_1} \Delta^1 \xrightarrow{f} \cancel{X}\ \mathcal{C}$$

is a homotopy from f to itself.

Now suppose $f, g, h \in \mathrm{Arr}(x,y)$ and that

$\sigma : \Delta^1 \longrightarrow \mathcal{C}$, $\sigma' : \Delta^1 \longrightarrow \mathcal{C}$ are homotopies from f to g and from f to h. Let

$$\sigma'' : \Delta^2 \longrightarrow \Delta^0 \xrightarrow{Y} \mathcal{C}$$

Then have

$$\begin{tikzcd}
\Lambda^3_1 \arrow[r, "{(\sigma'', \bullet, \sigma', \sigma)}"] \arrow[d] & \mathcal{C} \\
\Delta^3 \arrow[ur, dashed, "\tau"] &
\end{tikzcd}$$

and $d_1(\tau)$ is homotopy from g to h. If $h = f$ this shows ~~reflexi~~ symmetry and transitivity follows immediately from the previous argument.

$\square$

Prop ~~If $\{f\} \in \mathrm{Hom}$~~ Let $\mathrm{Hom}_{h\mathcal{C}}(x, y) = \mathrm{Arr}(x,y)/\sim$ and $\mathrm{Ob}\, h\mathcal{C} = \mathcal{C}_0$. Then this has a structure of a category by defining composition as

$$\begin{tikzcd}
& y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"] & & z
\end{tikzcd}$$

i.e let $\Lambda^2_1 \longrightarrow \mathcal{C}$ and extend to $\Delta^2 \longrightarrow \mathcal{C}$

Prop An object x of $\mathcal{C}$ is final/initial iff it is final/initial in $h\mathcal{C}$

~~Proof We ~~ ~~Only for final objects. We know x is final iff $\mathrm{Hom}^R_{\mathcal{C}}(y, x)$ is~~ ~~Kan complex~~ ~~contractible for all y.~~

Def An ~~inf~~ ∞-category is pointed if it has an object which is both initial and ~~&~~ final. Call this zero object. Zero morphism!

Examples (i) $h\,\mathrm{Top}$ is exactly the classical category of spaces modulo homotopy. Thus, $\emptyset$ is initial and any contractible space is final in Top

(ii) Top_* is defined similarly to Top:

n-simplices: $(X_0, \ldots, X_n)$ based spaces with

pointed maps $\left(h_{ij} : X_i \wedge \square^{j-i-1}_{\mathrm{Top}\,+} \longrightarrow X_j\right)$

satisfying coherence conditions.

Then $h\,\mathrm{Top}_*$ is the classical category of pointed space modulo pointed homotopy. Thus, any contractible space is a zero object.

(iii) Let R be a unital ring. For $n \in \mathbb{N}$, consider $P(\{1, \dots, n\})$ as a poset and let $N^{nd}(P(\{1, \dots, n\}))$ denote the non-degenerate simplices of $N(P(\{1, \dots, n\}))$. Let

$$\left(\Pi^{n}_{mod}\right)_q = \bigoplus_{N^{nd}(P(\{1, \dots, n\}))} R$$

with differentials induced by the maps on the nerve. Then ~~let~~ $(R\text{-mod})_q$ consists of

$$\cancel{(R\text{-mod})_q} \left(K^{(0)}, \dots, K^{(q)}\right)$$

together with $\left(h_{ij} : K^{(i)} \otimes \Pi^{j-i-1}_{mod} \to K^{(j)}\right)_{0 \leq i < j \leq q}$

and compatibility conditions

<u>Def</u> A triangle in a pointed ∞-category $\mathcal{C}$ is a diagram $\Delta^1 \times \Delta^1 \longrightarrow \mathcal{C}$ s.t.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d] & y \arrow[d] \\ 0 \arrow[r] & z \end{tikzcd}$$

It is a (co-)fiber sequence if it is a ~~pul~~ pushout/pullback

If $f: x \longrightarrow y$ is a morphism, a (co)fiber of f is

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ 0 \arrow[r] & Cf \end{tikzcd}$$
cofiber sequence

fiber sequence
$$\begin{tikzcd} Ff \arrow[r] \arrow[d] & x \arrow[d, "f"] \\ 0 \arrow[r] & y \end{tikzcd}$$

<u>Examples</u> (i) Let $f: X \longrightarrow Y$ be a map of pointed spaces

$$\begin{tikzcd} Ff \arrow[r] \arrow[d] & PY \arrow[d, "p_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
pullback

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ CX \arrow[r] & Cf \end{tikzcd}$$
Poshout.

Then $Ff \longrightarrow X \xrightarrow{f} Y$ is a fiber sequence

$X \xrightarrow{f} Y \longrightarrow Cf$ is a cofiber sequence

(ii) If $f : A_\bullet \longrightarrow B_\bullet$ is a chain map, then let $(Cf)_n = A_n \oplus B_{n-1}$ with

$$d(a,b) = (-d(a), d(b) - f(a))$$

Then $A_\bullet \xrightarrow{f} B_\bullet \longrightarrow Cf$ is both a fiber and cofiber sequence.

Def $\mathcal{C}$ is stable if,

(i) $\mathcal{C}$ is pointed

(ii) Every morphism has a fiber and cofiber

(iii) A triangle is a fiber sequence iff it is a cofiber sequence?

Examples !

Now let $\mathcal{C}^{\Sigma} \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$

$$\mathcal{C}^{\Omega} \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$$

be the subsimplicial sets generated by the diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0' \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

which are pushouts and pullbacks respectively. These are ∞-categories if $\mathcal{C}$ is.

Prop The evaluations (if $\mathcal{C}$ admits fibers and cofibers)

$$ev_{(0,0)}: \mathcal{C}^{\Sigma} \longrightarrow \mathcal{C}$$

$$ev_{(1,1)}: \mathcal{C}^{\Omega} \longrightarrow \mathcal{C}$$

have sections $s: \mathcal{C} \longrightarrow \mathcal{C}^{\Sigma}$, $\omega: \mathcal{C} \longrightarrow \mathcal{C}^{\Omega}$

and these are unique up to homotopy.

Def The suspension functor is the composite

$$\Sigma : \mathcal{C} \xrightarrow{s} \mathcal{C}^{\Sigma} \xrightarrow{ev_{(1,1)}} \mathcal{C}$$

Dually, the loop functor is the composite

$$\Omega : \mathcal{C} \xrightarrow{\omega} \mathcal{C}^{\Omega} \xrightarrow{ev_{(0,0)}} \mathcal{C}$$

Rem If $\mathcal{C}$ is stable, then $\mathcal{C}^{\Sigma} = \mathcal{C}^{\Omega}$ and one can check that Σ and Ω are inverse to eachother.

Lemma If $\mathcal{C}$ is stable, then $h\mathcal{C}$ admits coproducts and the natural map

$$\Sigma X \sqcup \Sigma Y \xrightarrow{\simeq} \Sigma(X \sqcup Y)$$

is an isomorphism

Prop ~~The homotopy category~~ If $\mathcal{C}$ is stable, $h\mathcal{C}$ is an additive category

Definition of the group structure on $\mathrm{Hom}_{h.c}(\Sigma X, Y)$

We can write ΣX as the colimit of

$$0 \leftarrow X \rightarrow X \leftarrow X \rightarrow \quad \rightarrow X \leftarrow X \rightarrow 0$$

and $\Sigma X \sqcup \_ \sqcup \Sigma X$ as colimit of

$$0 \leftarrow X \rightarrow 0 \leftarrow X \rightarrow 0 \quad \_\_ \rightarrow 0 \leftarrow X \rightarrow 0$$

Obtain a diagram

$$\begin{tikzcd}
0 \arrow[d] & X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] & X \arrow[l] \arrow[r] \arrow[d] & _ & X \arrow[l] \arrow[d] & X \arrow[l] \arrow[r] \arrow[d, "id"] & 0 \arrow[d, "0"] \\
0 & X \arrow[l] \arrow[r] & 0 & X \arrow[l] \arrow[r] & __ & 0 & X \arrow[l] \arrow[r] & 0
\end{tikzcd}$$

which yields map $\Sigma X \longrightarrow \Sigma X \sqcup \_ \sqcup \Sigma X$ after passing to colimits. Then can define a group law by

$$\mathrm{Hom}_{hc}(\Sigma X, Y) \times \_ \times \mathrm{Hom}_{hc}(\Sigma X, Y) \xrightarrow{\cong} \mathrm{Hom}(\Sigma X \sqcup \_ \sqcup \Sigma X, Y)$$
$$\longrightarrow \mathrm{Hom}(\Sigma X, Y)$$

If $f: \Sigma X \longrightarrow Y$, then it can be represented by a diagram

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "f''"'] & 0 \arrow[d] \\ 0' \arrow[r] & Y \end{tikzcd}$$

Then $-f$ is induced by

$$\begin{tikzcd} X \arrow[r, "f''"] \arrow[d] & 0' \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

and the zero map simply is $\Sigma X \longrightarrow 0 \longrightarrow Y$

~~Prop With this~~

This becomes abelian for $\mathrm{Hom}_{hC}(\Sigma^2 X, Y)$.

One can show this by defining multiplications by:

1) $\Sigma^2 X \longrightarrow \Sigma^2 X \vee \Sigma^2 X$

2) $\Sigma^2 X \longrightarrow \Sigma(\Sigma X \vee \Sigma X) \xrightarrow{\cong} \Sigma^2 X \vee \Sigma X$

and showing they are compatible according to Eckman - Hilton

Examples for stable and unstable categories

(i) $Ch^b(R\text{-mod})$ is stable

(ii) Top_* is not stable. For instance

$$S^1 \longrightarrow D^2 \longrightarrow D^2/S^1 \cong S^2$$

is a cofiber sequence, but it is not a fiber sequence ~~(LES of ho~~ If it were,

$$\cancel{\pi_2(D^2) \longrightarrow \pi_2(S^2) \longrightarrow \pi_1(S^1) \longrightarrow}$$

$$\cancel{\pi_3(D^2) \to \pi_3(S^2) \longrightarrow \pi_2(S^1) \longrightarrow \pi_2(D^2)}$$

$$\longrightarrow \underbrace{\pi_3(D^2)}_{=0} \longrightarrow \pi_3(S^2) \longrightarrow \underbrace{\pi_2(S^1)}_{=0} \longrightarrow \pi_2(D^2)$$

is exact, but $\pi_3(S^2) \neq 0$.

Similarly, $S^1 \longrightarrow S^3 \xrightarrow{\eta} S^2$

is a fiber sequence, but not a cofiber sequence, since this would induce

$$\tilde{H}^2(S^1) \longrightarrow \underbrace{\tilde{H}^2(S^3)}_{=0} \longrightarrow \underbrace{\tilde{H}^2(S^2)}_{\cong \mathbb{Z}} \longrightarrow \tilde{H}^3(S^1) \longrightarrow$$

$$=$$

<u>Def</u> A ~~triangulated~~ category $\mathcal{A}$ is triangulated if

(1) It is additive

(2) ~~The~~ It has an auto-equivalence

$$\Sigma : \mathcal{A} \longrightarrow \mathcal{A}$$

(3) It has a collection of distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

and following axioms are satisfied

(TR 1) (a) If $f : X \to Y$, then there is a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$

(b) Any triangle isomorphic to a distinguished one is itself distinguished

(c) $X \xrightarrow{id} X \longrightarrow 0 \longrightarrow \Sigma X$ is distinguished

(TR 2) $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$

is distinguished iff

$$Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \quad \text{is}$$

(TR 3) If the ~~hori~~ rows are distinguished triangles:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

(TR 4) If we have three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} \Sigma X$$

$$Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} \Sigma Y$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{w} Z/X \xrightarrow{d''} \Sigma X$$

there is a ~~trian~~ distinguished triangle

$$Y/X \xrightarrow{\varphi} Z/X \xrightarrow{\psi} Z/Y \xrightarrow{\Theta} Y/\Sigma X$$

s.t.

$$\begin{tikzcd}
X \arrow[rr, "g \circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, "v"] \arrow[dr, "w"'] & & Z/Y \arrow[rr, "\Theta"] \arrow[dr, "d'"'] & & Y/\Sigma X \\
& Y \arrow[ur, "g"'] \arrow[dr, "u"'] & & Z/X \arrow[ur, "\psi"'] \arrow[dr, "d''"'] & & \Sigma Y \arrow[ur, "\Sigma u"'] & \\
& & Y/X \arrow[ur, "\varphi"'] \arrow[rr, "d"] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$