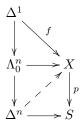
Exercise. Let $p:X\to S$ be a morphism of simplicial sets, and $f:\Delta^1\to X$ an edge, with source $x:\Delta^0\to X$ (that is, x is the composition of f with the canonical inclusion $\Delta^0\subseteq\Delta^1$). Show that the following are equivalent.

- (A) The induced map $X_{/f} \to X_x \times_{S_{/p(x)}} S_{/p(f)}$ is a trivial Kan fibration.
- (B) For every $n \geq 2$, and every commutative diagram



there is a dashed morphism making the diagram commutative.

Step a). Recall that a morphism is a trivial Kan fibration if and only if it has the left lifting property with respect to the morphisms $\partial \Delta^n \to \Delta^n$ for all $n \geq 0$. So (A) is equivalent to:

(A') For all $n \geq 0$ and every commutative diagram

(1)
$$\partial \Delta^{n} \xrightarrow{\qquad} X_{/f}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\Delta^{n} \xrightarrow{\qquad} X_{x} \times_{S/p(x)} S_{/p(f)}$$

there is a dashed morphism making the diagram commutative.

Step b). Recall that by the universal property of overcategories, the upper horizontal morphism in (1) corresponds to a unique commutative diagram

(2)
$$\Delta^1 \xrightarrow{f} X$$

where $\Delta^1 \subseteq \Delta^1 * \partial \Delta^n$ is the canonical inclusion (and conversely, such a diagram corresponds to a unique morphism $\partial \Delta^n \to X_{/f}$). Moreover, by the universal property of fibre products, the lower horizontal morphism in (1) corresponds to a unique commutative diagram

$$\Delta^n \longrightarrow S_{/p(f)}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$X_x \longrightarrow S_{/p(x)}$$

(where the lower morphism and right morphism are the canonical ones), and by the universal property of overcategories, such a diagram corresponds uniquely to a commutative diagram

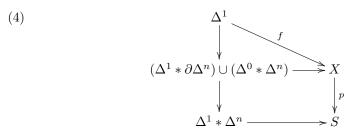
(3)
$$\Delta^{0} \xrightarrow{X} \Delta^{0} * \Delta^{n} \xrightarrow{X} X$$

$$\downarrow \qquad \qquad \downarrow^{p}$$

$$\Delta^{1} \xrightarrow{p(f)} S$$

where $\Delta^0 \subseteq \Delta^0 * \partial \Delta^n$, $\Delta^1 \subseteq \Delta^1 * \partial \Delta^n$, $\Delta^0 * \Delta^n \subseteq \Delta^1 * \Delta^n$ are the canonical ones.

Step c). Observe that the commutativity of (1) (without the dashed morphism) is equivalent to the requirement that (2) and (3) fit together into a commutative diagram



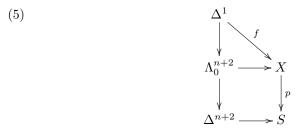
Step d). Observe that the inclusion

$$(\Delta^1 * \partial \Delta^n) \cup (\Delta^0 * \Delta^n) \subseteq \Delta^1 * \Delta^n$$

is canonically isomorphic to the inclusion

$$\Lambda_0^{n+2} \subseteq \Delta^{n+2}$$
.

(Indeed, $(\Delta^0 * \Delta^n) \subseteq \Delta^1 * \Delta^n$ is the inclusion of the face $d_1 \Delta^{n+2} \subseteq \Delta^{n+2}$. Moreover, $\partial \Delta^n$ is the union of the faces $d_i \Delta^n \subseteq \Delta^n$ for $0 \le i \le n$, and $\Delta^1 * d_i \Delta^n \subseteq \Delta^{n+2}$ is the inclusion of the face $d_{i+2} \Delta^{n+2} \subseteq \Delta^{n+2}$). Conclude that (4) is isomorphic to



 $Step\ e).$ Conclude that (1) admits a dashed morphism making the diagram commute if and only if the corresponding diagram



admits a dashed morphism making the diagram commute.