University of Amsterdam **Wintersemester 2020**
Infinity-categories

Exercise sheet 9: alternative models for $(\infty, 1)$-categories II

1 Let C be a differential graded category. Prove that, for any object $X \in C$, the identity morphism $\mathrm{id}_X \in \underline{\mathrm{Hom}}_C(X, X)_0$ is a 0-cycle; that is, that it satisfies $\partial(\mathrm{id}_X) = 0$. Hint: apply the Leibniz rule to the equation $\mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X$.

2 Let C be a differential graded category. Prove that the differential graded nerve $N^{\mathrm{dg}}(C)$ is a simplicial set; that is, that the action of morphisms in the simplex category Δ preserves the complicated condition on boundary of morphisms. Hint: this is a case distinction, and one case uses the previous exercise. See Kerodon Proposition 2.5.3.5 (Tag 00PQ) for details.

3 Let M be a model category. Show that the cofibrations and fibrations determine the weak equivalences. Hint: first show that they determine the trivial cofibrations and the trivial fibrations; then show that weak equivalences are exactly those maps which can be written as composition of a trivial cofibration followed by a trivial fibration.

4 Let M be a model category. In our definition (a variant of Quillen's original one, possibly due to Joyal-Tierney) we do not ask that the class of weak equivalences is stable under retracts. The goal of this exercise is to prove, following Joyal-Tierney, that this is automatic.

- Let w be a weak equivalence and f be a retract of w. Assume first that f is a fibration. Factor w as $v \circ u$ with u a trivial cofibration and v a trivial fibration (why is that possible?). Prove that f is then a retract of v. Conclude that f is a trivial fibration, and in particular is a weak equivalence.
- We now drop the assumption that f is a fibration. Factor $f = h \circ g$ with h a fibration and g a trivial cofibration. Because f is a retract of w, there exists a diagram

$$\begin{tikzcd}
\cdot \arrow[r, "r"'] \arrow[d, "g"'] & \cdot \arrow[r] \arrow[dd, "w"'] & \cdot \arrow[d, "g"'] \\
\cdot \arrow[d, "h"'] & & \cdot \arrow[d, "h"'] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

  where $\cdot$ denotes various objects of M (we don't need a notation for them) and the two horizontal compositions are identity morphisms. By considering the pushout of r and g and using the universal property of that pushout, complete this diagram into a 2*2 grid. Show that the pushout of a trivial cofibration along any map is a trivial cofibration (hint: saturated classes). Using this and the 2-out-of-3 property, prove that h is the retract of a weak equivalence. By the previous question, conclude that h is a weak equivalence. Deduce that f is as well, concluding the proof.

5 Let C be a category and W a collection of morphisms in C. We say that W satisfies the "two-out-of-six" property if for any three composable morphisms

$$\cdot \xrightarrow{u} \cdot \xrightarrow{v} \cdot \xrightarrow{w} \cdot,$$

if wv and vu are in W then so are u, v, w and wvu. Show that this implies the "two-out-of-three" property. Show that the class of weak equivalences in a model category satisfies the "two-out-of-six" property (hint: use Quillen's fundamental theorem).