Exercise sheet 8: alternative models for $(\infty, 1)$-categories

1 **Monoidal categories with one object.** I said something wrong about this in class, and the answer is actually very interesting, so let's try again as an exercise! Let V be a monoidal category with one object, which has to be the unit object 1_V of its tensor product $\otimes$.

- Let $M = \mathrm{Hom}_V(1_V, 1_V)$. The set M has two binary structures: $\circ : M \times M \to M$ coming from composition of morphisms, and $\otimes : M \times M \to M$ coming from tensor product of morphisms. Prove that both are monoid structures on M.
- Prove that, for all $a, b, c, d \in M$, we have

$$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d).$$

- Show that this implies that $\otimes = \circ$ and that this common monoid structure is commutative! (This is called the Eckmann-Hilton argument).
- Conclude that the category of monoidal categories with one object and monoidal functors is equivalent to the category of commutative monoids.

2 Let V be a monoidal category. Show that the functor $\mathrm{Hom}(1_V, -) : V \to \mathrm{Set}$ has a natural lax-monoidal structure. This is used to define the underlying category of any V-enriched category.

3 Let V be a symmetric monoidal category. Assume that for each $v \in V$, the functor $v \otimes - : V \to V$ admits a right adjoint $\underline{\mathrm{Hom}}(v, -) : V \to V$. Show that these assemble into a functor

$$\underline{\mathrm{Hom}}(-, -) : V^{\mathrm{op}} \times V \to V$$

such that there are isomorphisms

$$V(u \otimes v, w) \simeq V(u, \underline{\mathrm{Hom}}(v, w))$$

natural in $u, v, w \in V$. We then say that V is a closed symmetric monoidal category. Cartesian closed categories are examples of this. Show that a closed symmetric monoidal category V is "self-enriched": it is the underlying category of a V-category $\tilde{V}$ in a natural way (we saw this in class for cartesian closed categories).

4 Show that, if R is a commutative ring, the category of R-modules, equipped with its usual symmetric monoidal structure, is closed in the sense of the previous exercise.

5 Let V, W be monoidal categories and $F : V \leftrightarrows W : G$ be an adjunction. Assume given a lax monoidal structure on F (resp. an oplax monoidal structure on G). Construct an oplax monoidal structure on G (resp. a monoidal structure on F). This situation is sometimes called a lax/colax monoidal adjunction; the term "monoidal adjunction" is usually reserved for the more restricted situation where the left adjoint is actually monoidal (which often occurs in practice).

6 Show that the composite functor $\mathrm{Cat}_\Delta \xrightarrow{N_\Delta} \mathrm{sSet} \xrightarrow{\tau} Cat$ of the homotopy coherent nerve followed by the fundamental category functor is isomorphic to the homotopy category functor $h : \mathrm{Cat}_\Delta \to \mathrm{Cat}$.

7 Topological categories:

- Let $C \in \mathrm{Cat}_{\mathrm{Top}}$ be a topological category. Observe that the simplicial category $\mathrm{Sing}_*(C)$ is locally Kan and hence that $N_\Delta \mathrm{Sing}_*(C)$ is a quasicategory.
- Let CW be the category of CW-complexes. By using the fact that there exists cartesian closed subcategories of Top which contain all CW-complexes (the so-called convenient subcategories of Top which we discussed earlier, like compactly generated spaces), construct a topological enrichment $\widetilde{\mathrm{CW}} \in \mathrm{Cat}_{\mathrm{Top}}$ of CW.
- Show that geometric realisation $|-| : \mathrm{Kan} \to \mathrm{CW}$ can be upgraded a functor $\widetilde{\mathrm{Kan}} \to \mathrm{Sing}_*(\widetilde{\mathrm{CW}})$ of simplicial categories. More concretely, this means that one must construct, for X, Y Kan simplicial sets, a morphism $\mathrm{Fun}(X, Y) \to \mathrm{Sing}(\underline{\mathrm{Hom}}(|X|, |Y|)$ satisfying some properties; this can be done purely using the various adjunctions.
- One can show that the functor $\widetilde{\mathrm{Kan}} \to \mathrm{Sing}_*(\widetilde{\mathrm{CW}})$ from the previous question is a Dwyer-Kan equivalence (Hint if you want to show this: use the fact that if X is any simplicial set and Y is Kan, geometric realisation induces a bijection from the set of simplicial homotopy classes of maps $X \to Y$ to the set of homotopy classes of maps $|X| \to |Y|$). Using a theorem of Lurie mentioned in the course, conclude that the ∞-category $N_\Delta(\mathrm{Sing}_*(\widetilde{\mathrm{CW}}))$ is equivalent to the ∞-category Spc of spaces.