University of Amsterdam **Wintersemester 2020**
Infinity-categories

Exercise sheet 5: Infinity-categories II

1 Let $f : X \to Y$ be a morphism of simplicial sets; write $f^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$ for the induced morphism of opposite simplicial sets. Prove that if f is an inner fibration, f^{op} is an inner fibration. Prove that if f is a left fibration, then f^{op} is a right fibration.

2 Consider a pullback square of simplicial sets:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"'] \\ Y' \arrow[r, "\pi"'] & Y. \end{tikzcd}$$

Assume that π is a surjection. Show that if p' is an inner fibration, then p is an inner fibration.

3 Prove Lemma III.1.4 in the course: given $X \in \mathrm{sSet}$ and $C \in \mathrm{Cat}$ and any morphism $f : X \to NC$, prove that f is an inner fibration iff X is an ∞-category. Prove also that inner fibrations are stable under pullbacks (we will see this more generally in the lecture next week). Deduce the following: for any morphism $f : X \to Y$ of simplicial sets, then the following conditions are equivalent:

- f is an inner fibration.
- For all simplices $\sigma : \Delta^n \to Y$, the pullback $X \times_Y \Delta^n \to \Delta^n$ is an inner fibration.
- For all simplices $\sigma : \Delta^n \to Y$, the pullback $X \times_Y \Delta^n$ is an ∞-category.

4 Let $H^l := \Delta^2/\Delta^{\{0,1\}}$, i.e. the pushout of the diagram $\Delta^2 \hookleftarrow \Delta^{\{0,1\}} \to \Delta^0$. Show that H^l is not an ∞-category. Let $f : \Delta^1 \to H^l$ be the composite $\Delta^1 \overset{\langle 02 \rangle}{\to} \Delta^2 \overset{\pi}{\to} H^l$ where π is the quotient map. Show that f is an inner fibration by computing its base change along π, identifying it with a functor between nerves of categories, and applying the two previous exercises. This gives a simple example of an inner fibration (due to Alexander Campbell) whose target is not an ∞-category.

5 We have seen that the "homotopy category functor" from infinity-categories to categories preserves arbitrary products. The situation is more complicated for the "fundamental category" functor $\tau : \mathrm{sSet} \to \mathrm{Cat}$: show that the canonical map

$$\tau(\prod_{n \in \mathbb{N}} I^n) \to \prod_{n \in \mathbb{N}} \tau(I^n)$$

is not an isomorphism (or even equivalence) of categories. Hint: It is not surjective on morphisms; here is an explicit counter-example. Let f_n be the composite of the n-morphisms $0 \to 1, 1 \to 2, \ldots, n-1 \to n$ in $\tau(I^n)$. Show that $(f_n)_{n \in \mathbb{N}}$ which is a morphism from $(0)_{n \in \mathbb{N}}$ to $(n)_{n \in \mathbb{N}}$ in $\prod_{n \in \mathbb{N}} \tau(I^n)$ is not in the image.

On the other hand, τ does commute with finite products, but the proof is not so easy; we will see a proof later in the course (or see [Cisinksi, Lemma 3.3.13]).