University of Amsterdam **Wintersemester 2020**
Infinity-categories

Exercise sheet 4: Infinity-categories

1 Let C be an ∞-category. Let $\sigma : \Delta^1 \times \Delta^1 \to C$ be a morphism of simplicial sets. We call such a σ a commutative square in C. Show that σ induces a commutative square in the usual sense in the homotopy category hC.

2 Let C be an ∞-category and let $\bar{\sigma} : [1] \times [1] \to hC$ be a commutative square in hC. Show that there exists a commutative square σ in C such that $\bar{\sigma}$ is induced by σ. Show that σ is not necessarily unique.

3 Show by exhibiting an example that for any $n \geq 1$, the n-skeleton of an ∞-category (resp. a Kan complex) is not in general an ∞-category (resp. a Kan complex).

4 Show that there exists pushout (resp. coequalizer) diagrams of simplicial sets where every object is a ∞-category but the colimit in sSet is not an ∞-category (hint: for both, you can look at the spine I^2).

5 Let $(C_\alpha)_{\alpha \in J}, (D_\alpha)_{\alpha \in J}$ be two families of ∞-categories indexed by the same set J. Suppose given, for each α, a categorical equivalence $F_\alpha : C_\alpha \to D_\alpha$. Construct a categorical equivalence $\prod_{\alpha \in J} C_\alpha \to \prod_{\alpha \in J} D_\alpha$ (resp. $\coprod_{\alpha \in J} C_\alpha \to \coprod_{\alpha \in J} D_\alpha$). Why does this construction not work for any (co)limit of quasicategories?

6 Let C be a category which admits filtered colimits. An object $c \in C$ is called *compact* if the corepresentable functor $C(c, -) : C \to \mathrm{Set}$ commutes with filtered colimits. In the course, we showed (in the proof that filtered colimits of ∞-categories are ∞-categories) that simplicial sets with finitely many non-degenerate simplices are compact in sSet. Show that an object in Set is compact iff it is finite. Show that if R is a commutative ring and C is the category of R-modules, then an R-module is compact iff it is of finite presentation (i.e. can be written as cokernel of a map $R^m \to R^n$ for $m, n \in \mathbb{N}$).

7 (More difficult) Show that, for any $n \in \mathbb{N}$, the n-th coskeleton $\mathrm{cosk}_n(C)$ of an ∞-category C (see Exercise 2.2) is an ∞-category (hint: use that cosk_n is the right adjoint of sk_n and compute the skeleta of horn and standard simplices). By adjunction, there is a canonical map $C \to \mathrm{cosk}_2(C)$. Show that the induced functor $hC \to h\mathrm{cosk}_2(C)$ is an equivalence. Deduce from this that the natural map $C \to Nh(C)$ factors through the natural map $C \to \mathrm{cosk}_2(C)$, inducing a map $\mathrm{cosk}_2(C) \to Nh(C)$ (hint: this requires playing with the adjunction between h and N). This map is not an isomorphism in general, but one can prove that it is an equivalence of ∞-categories! See `https://mathoverflow.net/questions/306470/what-is-the-coskeleton-tower-of-a-quasi-category` for a discussion (although one which uses many concepts we have not seen so far).