University of Amsterdam **Wintersemester 2020**
Infinity-categories

Exercise sheet 3: Kan complexes and nerves

1 Read the definitions and the statements (not the proofs!) of results in Kerodon section 1.1.5 on directed graphs, and prove the results there (1.1.5.7-9). This explains how to fully embed directed graphs into simplicial sets, as the subcategory of 1-skeletal (or 1-dimensional) simplicial sets.

2 Prove that the functor $\pi_0 : \mathrm{sSet} \to \mathrm{Set}$ preserves all colimits (Hint: use the description $\pi_0(X) = \mathrm{Colim}(X)$. Show then that π_0 also preserves finite products. More precisely, since $\pi_0(\Delta^0) = \{*\}$ (so that π_0 preserves final objects), this amounts to show that the canonical map $\pi_0(X \times Y) \to \pi_0(X) \times \pi_0(Y)$ induced by the projections is a bijection.

3 Let C be a category. The classifying space of C is by definition the geometric realisation $|N(C)|$ of its nerve. Prove that if $F : C \to D$ is a left (or right) adjoint, then $|N(F)|$ is an homotopy equivalence. Deduce that if C admits either an initial or a final object, then $|N(C)|$ is contractible.

4 Let $n \geq 2$. Show that $\partial\Delta^n$, Λ^n_k and I^n are not in the essential image of the nerve functor and are also not Kan complexes. Are they infinity-categories? Hint: almost none of them are.

5 Let X be a topological space. Describe $\tau\mathrm{Sing}(X)$, the fundamental category of the singular complex of X: show that it is a groupoid, equivalent to the fundamental groupoid of X (if you don't know what the fundamental groupoid is, show that the group of automorphisms of objects in X correspond to the fundamental groups of X at various base points).

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