University of Amsterdam **Wintersemester 2020**
Infinity-categories

Exercise sheet 2: Presheaves and simplicial sets 2

1 Prove (at least some of) the simplicial identities!

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- Let $F : C \to D$ be a functor between small categories. There is an induced functor $F^* : \mathrm{PSh}(D) \to \mathrm{PSh}(C)$ given by precomposition by F. Prove that this functor has a left adjoint, which we will denote by $F_!$ (Hint: use the free cocompletion property to construct a candidate functor and check it is indeed a left adjoint to F^*).
- Prove that F^* also admits a right adjoint, that we will denote by F_* (Hint: show that F^* preserves colimits). There is a formula for F_* in terms of limits, but we will not need it; see Riehl's "Category theory in context" Section 6.2.
- For $n \geq -1$, let $\Delta_{\leq n}$ be the full subcategory of the simplex category Δ whose objects are $[i]$ for $i \leq n$ (in particular $\Delta_{\leq -1}$ is the empty category). We have the inclusion functor $i_n : \Delta_{\leq n} \to \Delta$. Prove that the construction of the n-th skeleton $\mathrm{Sk}_n(-)$ given in the lecture is functorial, and that there is a natural isomorphism

$$\mathrm{Sk}_n(-) \simeq (i_n)_!(i_n)^*.$$

  Deduce from all of this that $\mathrm{Sk}_n(-)$ has a right adjoint, the **coskeleton** functor $\mathrm{Cosk}_n(-) := (i_n)_*(i_n)^*$.

3 Show that, for any simplicial set X, geometric realisation induces a bijection $\pi_0(X) \to \pi_0(|X|)$ (where the first π_0 is the set of connected components of a simplicial set, and the second π_0 is the set of path-connected components of a topological space).

4 Show that, for a Kan complex X, the homotopy relation on X_0 defining $\pi_0(X)$ is indeed an equivalence relation.

5 Prove that discrete simplicial sets (in the sense of Exercise 6 of Sheet 1) are Kan complexes.

6 Prove that trivial Kan fibrations are Kan fibrations (and consequently that contractible Kan complexes are Kan complexes).

7 Prove that products and (more difficult) coproducts of Kan complexes are Kan complexes.

8 (Difficult) Prove that if $G_\bullet$ is a simplicial group, that is, a group object in the category of simplicial set, then the underlying simplicial set of $G_\bullet$ is a Kan complex. (For a solution, see Kerodon Proposition 1.1.9.9 (tag 00MG)).