University of Amsterdam **Wintersemester 2020**
Infinity-categories

Exercise sheet 1: Presheaves and simplicial sets

All categories are assumed locally small.

1 Given two functors $F : D \to C$ and $G : E \to C$, show that there is a category, called the **comma category** $F \downarrow G$, which as as objects triples $(d \in D, e \in E, f : Fd \to Ge)$ and as morphisms $(d, e, f) \to (d', e', f')$ a pair of morphisms $(h : d \to d', k : e \to e')$ such that the "obvious" square commutes (figuring out the square is part of the exercise!).

2 Show that the category of elements of a presheaf $F : C^{\mathrm{op}} \to \mathrm{Set}$ is isomorphic to the comma category $y \downarrow \widetilde{F}$ where y is the Yoneda embedding and $\widetilde{F}$ is the functor $* \to \mathrm{PSh}(C)$ from the one-point category corresponding to F.

3 Show that $F : C^{\mathrm{op}} \to \mathrm{Set}$ is representable if and only if the category of elements $\int F$ has a terminal object.

4 Recall that there is a functor

$$\mathrm{Hom} : C^{\mathrm{op}} \times C \to \mathrm{Set}, (X, Y) \to C(X, Y).$$

Explain how to view it as a presheaf on $C \times C^{\mathrm{op}}$ and describe its category of elements. It is called the "twisted arrow category".

5 Let Δ denote the simplex category, and $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. What is the limit of X, seen as a functor? Harder: what is its colimit? The resulting set is called the set $\pi_0(X)$ of connected components of X. We will study $\pi_0(X)$ in more detail in the course.

6 A simplicial set X is **discrete** if every morphism in Δ induces a bijection on the sections of X. Show that the functor "evaluation at [0]" induces an equivalence between the full subcategory of discrete simplicial sets and the category of sets.