Prop 55: Let $K \in sSet$, $C \in Cat_\infty^1$ and

$\tilde{p} : \begin{cases} K^{\triangleright} \\ K^{\triangleleft} \end{cases} \longrightarrow C$. Write $p = \tilde{p}|K$.

Then $\tilde{p}$ is a $\begin{cases} \text{colimit} \\ \text{limit} \end{cases}$ diagram

$$\Leftrightarrow \begin{cases} C_{\tilde{p}/} \longrightarrow C_{p/} \\ C_{/\tilde{p}} \longrightarrow C_{/p} \end{cases} \text{ is a trivial fibration.}$$

proof: See exercise sheet 12, or [Rezk, 28.7] $\square$

Def 56: Let $C \in Cat_\infty^1$ and $K \in sSet$

* C admits all (co)limits of shape K if every $p: K \to C$ has a (co)limit.

* C is (co)complete if C admits (co)limits of shapes all (small) simplicial sets.

* C admits finite (co)limits if C admits (co)limits of shapes all simplicial sets with finite

finitely many non-degenerate simplices. } simplicial sets

o o o

* Let $F : C \to D$ be a functor between ∞-cat.

We say that F preserves limits of shape K if for all $p : K \longrightarrow C$, the functor

$$C_{/p} \longrightarrow D_{/Fp}$$ sends terminal objects to terminal objects.

o o o

Def 57: Let $\delta : K' \to K$ be a morphism in sSet. We say that δ is final (resp. cofinal) if for every functor $p : K \longrightarrow C$ with C ∞-category, the induced functor

$$C_{/p} \longrightarrow C_{/p\delta}$$

$$\left(\text{resp. } C_{p/} \longrightarrow C_{p\delta/}\right)$$

sends terminal objects to terminal objects (resp. initial objects to initial objects).

It is possible to establish then many results parallel to the 1-categorical story. Here is a sample, without proofs for lack of time (also, some are probably easier to prove with other definitions of (co)limits).

Prop 58: [HTT, Cor 4.4.2.4 / Prop 4.4.2.6] Let $\mathcal{C}$ be an ∞-category. Then

- $\mathcal{C}$ admits finite $\left|\begin{array}{l}\text{colimits}\\ \text{limits}\end{array}\right.$ iff it admits $\left|\begin{array}{l}\text{pushouts}\\ \text{pullbacks}\end{array}\right.$ and has $\left|\begin{array}{l}\text{an initial}\\ \text{a terminal}\end{array}\right.$ object.

- $\mathcal{C}$ is $\left|\begin{array}{l}\text{cocomplete}\\ \text{complete}\end{array}\right.$ iff it admits $\left|\begin{array}{l}\text{pushouts}\\ \text{pullbacks}\end{array}\right.$ and arbitrary $\left|\begin{array}{l}\text{coproducts}\\ \text{products}\end{array}\right.$.

Prop 59: [HTT, Cor 4.4.2.5, Prop 4.4.2.7]

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between ∞-categories

- Assume that C admits finite $\left|\begin{array}{l}\text{colimits}\\ \text{limits}\end{array}\right.$

Then F preserves all finite $\left|\begin{array}{l}\text{colimits}\\ \text{limits}\end{array}\right.$ iff

F preserves $\left|\begin{array}{l}\text{pushouts}\\ \text{pullbacks}\end{array}\right.$ and $\left|\begin{array}{l}\text{initial}\\ \text{terminal}\end{array}\right.$ objects.

- Assume that C is $\left|\begin{array}{l}\text{cocomplete}\\ \text{complete}\end{array}\right.$. Then F

preserves all $\left|\begin{array}{l}\text{colimits}\\ \text{limits}\end{array}\right.$ <u>iff</u> F preserves $\left|\begin{array}{l}\text{pushouts}\\ \text{pullbacks}\end{array}\right.$

and $\left|\begin{array}{l}\text{coproducts}\\ \text{products}\end{array}\right.$.

Prop 60: [HTT, 4.2.3.14]

Let K be a simplicial set. There exists a 1-category I and a (co)final functor $p: N(I) \longrightarrow K$.

In particular, let C be an ∞-category. Then C is (co)complete <u>iff</u> for all small 1-categories I and functors $p: N(I) \to C$,

p admits a (co)limit.

Prop 61: [HTT, 4.1.1.10; ?]

Categorical equivalences are final, cofinal and preserve all limits and colimits.

- Let us briefly discuss the compatibility of this theory of (co)limits with the more classical theory of homotopy (co)limits. There are different formulations, let's focus on the simplest case.

Thm 62: Let $\mathcal{M}$ be a model category. I a small category.

Assume that $\mathcal{M}$ is combinatorial (technical condition, often satisfied). Then:

- $\mathrm{Fun}(I, \mathcal{M})$ admits a projective model structure such that
  - weak equivalences are objectwise
  - fibrations are objectwise.

which implies that the adjunction:

$$\operatorname{colim}_I : \mathrm{Fun}(I, \mathcal{M}) \rightleftarrows \mathcal{M} : \mathrm{const}$$

is a Quillen adjunction.

- $\mathrm{Fun}(I, \mathcal{M})$ admits an injective model structure such that : · weak equivalences are objectwise

· cofibrations are objectwise

which implies that the adjunction:

$$\mathrm{const} : \mathcal{M} \rightleftarrows \mathcal{M}^{I} : \lim_{I}$$

is a Quillen adjunction.

Def 63: With the same notations:

- The **homotopy colimit functor** is $\mathbb{L}\operatorname{colim}_{I}$ (derived using the projective model structure).
- The **homotopy limit functor** is $\mathbb{R}\lim_{I}$ (derived using the injective model structure).

Rmk 64 In particular cases, other model structures on $\mathrm{Fun}(I, M)$ with more control on (co)fibrations can be used. In particular, when I is a Reedy category (like $\begin{smallmatrix} \bullet & \to & \bullet \\ \downarrow & & \\ \bullet & & \end{smallmatrix}$, $\bullet \rightrightarrows \bullet$, $\mathbb{N}$, $\mathbb{N}^{op}$, Δ, Δ^{op} ...) there are Reedy model structures on $\mathrm{Fun}(I, \mathcal{M})$.

- If $\mathcal{M}$ is also a simplicial model category and I a simplicial category, then $\mathrm{Fun}_{sSet}(I, \mathcal{M})$ can also be equipped with appropriate model structures.

- Here is the comparison theorem.

Thm 65: (special case of [HTT, Thm 4.2.4.1] Cor 4.2.4.8)

Let $\mathcal{M}$ be a simplicial combinatorial model category.

1) Let I be a small category.

Let $F : I \longrightarrow \mathcal{M}$ be a functor.

Hyp: F factors through $\mathcal{M}_{cf}$.

(this can be arranged up to objectwise weak equivalence in $\mathrm{Fun}(I, \mathcal{M})$)

Let $\tilde{F} : N(I) \longrightarrow N_\Delta(\mathcal{M}_{cf})$ be the induced functor of ∞-categories.

Then $\tilde{F}$ admits a colimit and a limit (in the ∞-categorical sense) and we have

$$\begin{cases} \operatorname{colim}(\breve{F}) \simeq \mathbb{L}\operatorname{colim}(F) \\ \lim(\breve{F}) \simeq \mathbb{R}\lim(F) . \end{cases}$$

2) The ∞-category $N_\Delta(\mathcal{M}_{cf})$ is complete and cocomplete. $\square$

Rmk 66 Part 2) follows from 1) together with Prop 60 and a rectification result: any diagram $N(I) \to N_\Delta(\mathcal{M}_{cf})$ can be "rectified", up to isomorphism in $\mathrm{Fun}(N(I), N_\Delta(\mathcal{M}_{cf}))$, to a functor $I \longrightarrow \mathcal{M}_{cf}$.

Example 67:

- Let $\mathcal{M}$ be a simplicial combinatorial model category. Then the associated ∞-category $N_\Delta(\mathcal{M}_{cf})$ is complete and cocomplete.

In particular, Spc , $\mathcal{D}^+(\mathcal{A})$ (for $\mathcal{A}$ abelian cat. with enough injectives) and many other important examples are complete and cocomplete.

- Finally, we try to compute some limits and colimits in Spc "by hand", using simplicial homotopy theory.

- First, some generalities about (co)limits in homotopy coherent nerves. Let $\mathcal{C}$ be a locally Kan simplicial category, K a simplicial set and $p : K \longrightarrow N_\Delta(\mathcal{C})$ a functor.

  Recall the adjunction

$$\mathrm{Path}[-] : \mathrm{sSet} \rightleftarrows \mathrm{Cat}_\Delta : N_\Delta$$

- Using this, we see that a colimit of p is eq. to a functor $\hat{q} : \mathrm{Path}[K^\triangleright] \longrightarrow \mathcal{C}$ in Cat_Δ.

extending the adjoint $\mathrm{Path}[K] \xrightarrow{q} \mathcal{C}$ of p, and such that for all $n \geq 1$, a lift exists in

$$\begin{tikzcd}
\mathrm{Path}[K^{D}] \arrow[r] \arrow[rr, bend left, "\hat{q}"] & \mathrm{Path}[K * \partial\Delta^n] \arrow[r] \arrow[d] & \mathcal{C} \\
& \mathrm{Path}[K * \Delta^n] \arrow[ur, dashed] &
\end{tikzcd}$$

So it is a matter of understanding those simplicial path categories in special cases.

- Recall that we already determined the initial and terminal objects of Spc. (Prop 50).

<u>Products and coproducts:</u>

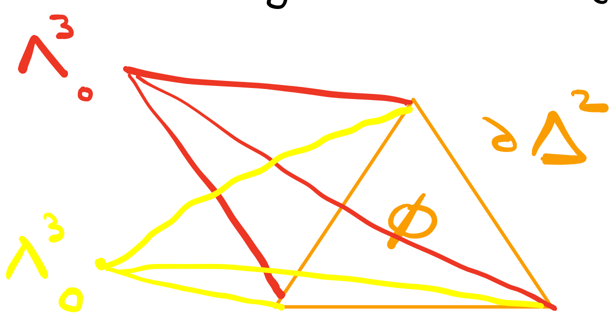
Let S be a set and $p : cS \longrightarrow \mathrm{Spc}$, which is equivalent to giving an S-indexed family $\{X_s\}_{s\in S}$ of Kan complexes.

By definition, $\begin{cases} \text{a coproduct of } \{X_s\}_{s\in S} \text{ is a colimit of } p \\ \text{a product of } \{X_s\}_{s\in S} \text{ is a limit of } p. \end{cases}$

Prop 68: The (co)product in the 1-category of Kan complexes is a (co)product of $\{X_s\}_{s\in S}$.

proof: We have
$$cS * \partial\Delta^n = \Big(\coprod_{s\in S} \Delta^0\Big) * \partial\Delta^n$$
in $sSet_{\partial\Delta^n/}$
$$= \coprod_{s\in S} \left(\partial\Delta^n \to \Delta^0 * \partial\Delta^n\right)$$

which as a simplicial set is a pushout of copies of $\Delta^0 * \partial\Delta^n = \Lambda^{n+1}_0$ glued along $\partial\Delta^{\{1,\dots,n\}}$

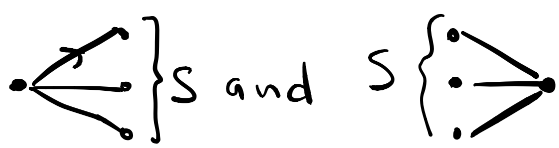


and similarly $cS * \Delta^n$ is a pushout of copies of Δ^{n+1} glued along $\Delta^{\{1,\dots,n\}}$.

- Since Path[-] is colimit-preserving, we can also describe Path[$cS * \partial\Delta^n$] and Path[$cS * \Delta^n$] quite concretely in terms of Path[Λ^{n+1}_0] and Path[Δ^{n+1}], which we already know quite well.

  The same discussion applies dually to Path[$\partial\Delta^n * cS$] $\subseteq$ Path[$\Delta^n * cS$].

- Even more simply, Path[$(cS)^{\triangleleft}$] and Path[$(cS)^{\triangleright}$] are constant simplicial categories



.

<u>**Products**</u>: Let check that $\prod X_s$ (in sSet) is a product of $\{X_s\}$ in the ∞-cat. sense.

$$\mathrm{Path}[(cS)^{\triangleleft}] \xrightarrow{\hat{q}} \widetilde{\mathrm{Kan}}$$

is simply given by the collection of maps.

$\left\{ \prod X_s \longrightarrow X_s \right\}_{s \in S}$. A diagram

$$\begin{tikzcd}
\mathrm{Path}[(cS)^{\triangleleft}] \arrow[r] \arrow[rr, bend left, "\hat{q}"] & \mathrm{Path}[\partial\Delta^n * cS] \arrow[r, "\circledast"] \arrow[d] & \widetilde{\mathrm{Kan}} \\
 & \mathrm{Path}[\Delta^n * cS] \arrow[ur, dashed] &
\end{tikzcd}$$

is determined by the map $\circledast$, which

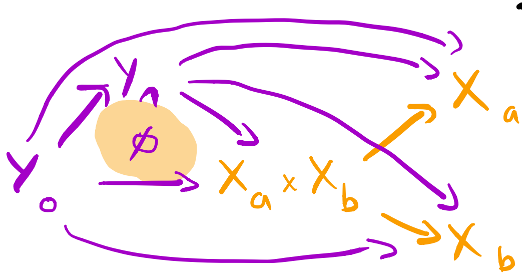
is given by a collection $\left\{ \mathrm{Path}[\Lambda^{n+1}_{n+1}] \longrightarrow \widetilde{\mathrm{Kan}} \right\}_{s \in S}$

which all send $\{n, n+1\}$ to the projection

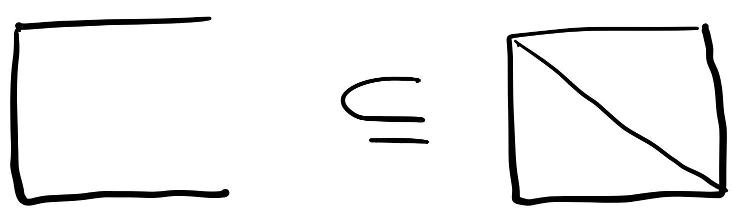
$\prod X_s \longrightarrow X_s$ and coincide on $\mathrm{Path}[\Delta^{\{0,\dots,n\}}]$

<u>Ex</u> for $S = \{a, b\}$;

$n = 1$



We have to extend to $\{ \mathrm{Path}[\Delta^{n+1}] \to \widetilde{Kan} \}_{s \in S}$ with the same properties. The point is that we can do this separtely for each S, and the only difference between the two simplicial categories $\mathrm{Path}[\Lambda^{n+1}_{n+1}]$ and $\mathrm{Path}[\Delta^{n+1}]$ is the mapping space $0 \longrightarrow n+1$



For $n \geq 2$, we can fill in as we like.

For $n = 1$, we have

$$\begin{tikzcd}
 & & X_a \\
Y_0 \arrow[r, dashed] \arrow[urr, bend left] \arrow[drr, bend right] & X_a \times X_b \arrow[ur] \arrow[dr] & \\
 & & X_b
\end{tikzcd}$$

and we fill in with the product map given by the universal property of the produkt in sSet.

Coproducts The argument is the same, dualized. $\square$

Pushouts and pullbacks

Prop 69: Let $p : \Lambda^2_0 \longrightarrow \mathrm{Spc}$ be a diagram:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \\ Y & \end{tikzcd}$$

Assume that either of f or g is a monomorphism.

(this can be achieved up to homotopy, by using the WFS $\underset{\text{mono}}{\hookrightarrow} \cdot \underset{\text{trivial fib.}}{\overset{\sim}{\twoheadrightarrow}}$ in sSet)

Then the pushout diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \arrow[d, "\tilde{f}"] \\ Y \arrow[r, "\tilde{g}"'] & Y \cup_X Z \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

in the 1-category sSet is a pushout

in the ∞-category Spc.

"proof" Apply the comparison Theorem above to $\mathcal{M} = sSet_{\text{Kan-Quillen}}$, combined with [HTT, § A.2.4]. $\square$

Dually, with the same proof:

Prop 70: Let $p : \Lambda^2_2 \longrightarrow \mathrm{Spc}$ be a diagram

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}.$$

Assume that either f or g is a Kan fibration.

(this can be achieved up to homotopy by using the WFS $\underset{\text{anodyne}}{\overset{\sim}{\hookrightarrow}}$ $\underset{\text{Kan fib.}}{\twoheadrightarrow}$ in sSet)

Then the pullback diagram

$$\begin{tikzcd} Y \times_X Z \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "g"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

in the 1-category sSet is a pullback in the ∞-category Spc. $\square$

These results also apply in the ∞-category of pointed spaces

$$\mathrm{Spc}_* := \mathrm{Spc}_{\Delta^0/} \simeq N_\Delta\left(\widetilde{\mathrm{Kan}}_{\Delta^0/}\right).$$

There, we have two fundamental examples.

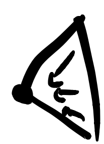
Example 71. Let $p : \Lambda^2_0 \longrightarrow \mathrm{Spc}_*$ given by

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \\ * & \end{tikzcd}.$$

We have another diagram $q : \Lambda^2_0 \longrightarrow \mathrm{Spc}_*$ given by

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X^{\triangleleft} \\ X^{\triangleleft} & \end{tikzcd}$$

with $X^{\triangleleft}$ the cone (in Spc, $X^{\triangleleft} \simeq X^{\triangleright}$!)

$|X^{\triangleleft}| \simeq$  X

p and q are naturally equivalent,

because $X^{\triangleleft}$ is a contractible Kan complex,

"so" they have the same colimit.

$X \longrightarrow X^{\triangleleft}$ is mono, so now we

apply Prop 69 and see that

$$* \coprod_{X}^{\infty} * \simeq X^{\triangleleft} \underbrace{\coprod_{X}}_{\text{on sSet}} X^{\triangleleft}$$

$$\simeq \Sigma X \text{ (suspension of } X\text{)}$$

In particular,

$$\left| * \coprod_{X}^{\infty} * \right| \simeq_{\text{hmtpy}} \Sigma |X|$$

- By a similar argument, one gets

$$* \times_{X}^{\infty} * \simeq \Omega X \text{ (loop space of } X\text{)}.$$

$$\left| * \times_{X}^{\infty} * \right| \simeq_{\text{hmpty.}} \Omega |X|$$

Def : $\mathcal{C}$ ∞-category is pointed if there is a zero object, i.e both initial and termial.

Ex : • Spc_* ∞-category of pointed spaces

• $\mathcal{D}^+(\mathcal{A})$ ∞-derived category.

• Non-example : Spc.

Def : Let $\mathcal{C}$ be a pointed ∞-cat with zero object 0. A (co)fiber sequence is a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circledast"] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

such that

• this is a pushout square (cofiber)

• —— pullback square (fiber)

We say that g (or Z) is the cofiber of f

—————— f (or X) —— fiber of g.

Def A pointed ∞-category is stable if every morphism admits a fiber and a cofiber, and a diagram (*) is a cofiber sequence iff it is a fiber sequence.

Ex: $\cdot\ \mathcal{D}^+(\mathcal{A})$ is stable.

Non-ex: $\mathcal{S}pc_*$

Rmk:

- $\mathcal{C}$ stable $\Rightarrow$ $h\mathcal{C}$ has a canonical structure of triangulated category.
- $\mathcal{S}pc_*$ admits a canonical stabilisation, the ∞-category of spectra.

6) Isomorphisms and the Joyal extension and lifting theorems

- Let C be an ∞-category and $f: x \to y$ a morphism. Recall that f is an isomorphism <u>iff</u> $R(f) \in RC(x,y)$ is. Recall the core $\mathrm{Core}(C) = C^{\simeq}$ is the (non-full) subcategory spanned by isomorphisms in C.

- We are going to study in more details isomorphisms in ∞-categories, and prove a fundamental characterisation of them due to Joyal. This will in particular imply that quasigroupoids are Kan complexes.
- We first need some preliminaries about conservative functors and isofibrations.

Def 72: Let C, D be ∞-categories. A functor $F: C \longrightarrow D$ is conservative if for all $f \in C_1$, $F(f)$ iso. $\Leftrightarrow$ f iso.

Rmk: · The corresponding notion in 1-category theory is also very useful.

Prop 73: Left / right fibrations between ∞-categories are conservative.

proof: · Let $F : C \to D$ be a right fibration between ∞-categories.

· Let $f : x \to y$ be a morphism in C such that $F(f)$ is an isomorphism.

$$\begin{tikzcd} & 1 \arrow[dr, "f"] & \\ 0 \arrow[rr, equal] & & 2 \end{tikzcd}$$

Let $a : \Lambda^2_2 \longrightarrow C$ with $a_{12} = f$ and $a_{02} = id_y$.

Since $F(f)$ is an isomorphism, there exists

$$\begin{tikzcd} & \cdot \arrow[dr, "F(f)"] & \\ \cdot \arrow[ur, dashed] \arrow[rr, equal] & & \cdot \end{tikzcd}$$

$b : \Delta^2 \longrightarrow C$ such that $b_{12} = F(f)$ and $b_{02} = id_{F(y)}$.

We get a commutative diagram:

$$\begin{tikzcd} \Lambda^2_2 \arrow[r, "a"] \arrow[d] & C \arrow[d, "F"] \\ \Delta^2 \arrow[r, "b"] \arrow[ur, dashed, "s"] & D \end{tikzcd}$$

and we have a lift s since F is a right fibration.

Then $g = s|_{\Delta^{\{0,1\}}}$ is such that $[f] \circ [g] \underset{\text{in } hC}{=} [id_y]$

$F(g)$, being an inverse of $F(f)$ is an iso;

by the same argument, there exists $h : x \to y$ such that $[g] \circ [h] = [id_x]$. This implies that $[f] = [h]$ and $[g]$ are isos, hence f is an iso. $\square$

Cor 74: (Slices of ∞-quasigroupoids are ∞-quasigroupoids)

Let C be a quasigroupoid and $x \in C_0$. Then $C_{x/}$ and $C_{/x}$ are also quasigroupoids.

proof: The restriction map $C_{x/} \longrightarrow C$ (resp. $C_{/x} \longrightarrow C$) is a left (resp. right) fibration, hence conservative by Prop 73. When $F : D \to D'$ is a conservative functor with target a quasi groupoid, then D is also a quasi-groupoid. $\square$

Def 75: Let C, D be ∞-categories.

A functor $F : C \to D$ is an *isofibration* if it is an inner fibration and F "lifts isomorphisms": for any $c \in C_0$ and $g : F(c) \xrightarrow{\sim} d'$ isomorphism, there exists $c' \in C_0$ and $f : c \xrightarrow{\sim} c'$ such that $F(f) = g$.

Equivalently, in terms of cores, we can solve the lifting problem:

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d, hook] & \mathrm{Core}(C) \arrow[r] \arrow[d, "\mathrm{Core}(F)"] & C \arrow[d, "F"] \\
\Delta^1 \arrow[r, "g"'] \arrow[ur, dashed, "f"] & \mathrm{Core}(D) \arrow[r] & D
\end{tikzcd}$$

- Isofibrations of 1-categories are defined similarly.

Prop 76: Let $F : C \to D$ be an inner fibration of simplicial sets. Then

F is an isofibration $\iff$ RF is an isofibration.
($\iff$ NRF ————)

proof: $\Rightarrow$ is clear since isomorphisms are detected by $h(-)$.

$\Leftarrow$: Let $g : F(c) \to d'$ be an isomorphism in D. Since RF is an isofibration, there exists an isomorphism $f' : c \to c'$ such that

$[F(f')] = [g]$. Choose $b : \Delta^2 \to D$ which exhibits this as a right homotopy:

$$\begin{tikzcd}
 & d' \arrow[dr, equal] & \\
F(c) \arrow[ur, "F(f')"] \arrow[rr, "g"'] \arrow[rr, phantom, bend left=20, "b"] & & d'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & {} \arrow[dr, equal, "\mathrm{id}_{c'}"] & \\
{} \arrow[ur, "f'"] & & {}
\end{tikzcd}$$

Let $a : \Lambda^2_1 \to C$ be the map with $a_{01} = f'$ and $a_{12} = \mathrm{id}_{c'}$.

We have a commutative diagram:

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r, "a"] \arrow[d] & C \arrow[d, "F"] \\
\Delta^2 \arrow[r, "b"'] \arrow[ur, dashed, "s"] & D
\end{tikzcd}$$

where a lift s exists since F is an inner fibration.

Then $f := s_{02}$ is a lift of g, and is an iso because $[f] = [f']$. $\square$

This implies the notion of isofibration is more symmetrical than it seems:

Cor 77: Let $F : C \to D$ be an isofibration between ∞-categories. Then we have:

for any $c \in C_0$ and $g' : d' \xrightarrow{\sim} F(c)$,

there exists $c' \in C_0$ and $g' : c' \xrightarrow{\sim} c$ such that $F(g') = g'$.

proof: This is clear for isofibrations between 1-categories, and the result then follows from the previous Proposition. $\square$

Prop 78: Left / right fibrations between ∞-categories are isofibrations.

proof: Let $F : C \to D$ be a right fibration with C, D ∞-categories. Let g be an iso in D.

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & C \arrow[d, "F"] \\ \Delta^1 \arrow[r, "g"'] \arrow[ur, dashed, "g"] & D \end{tikzcd}$$

The lift exists because F is a right fibration. (+ use Prop 77)

Now f is a lift of g, and since F is conservative by Prop , f is also an iso. $\square$

Rmk 79 Remember that the Joyal model structure on sSet is the unique model structure with:

- W = categorical equivalences.
- Cof = monomorphisms.

It is not too difficult to construct using the small object argument (see [Rezk, first part of Thm 45.8]). What is more difficult is to prove that

- fibrant objects are quasicategories.
- fibrations between quasicategories are precisely the isofibrations.

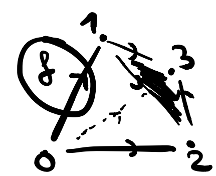
The proof (see [Rezk, second part of Thm 45.8])
Cisinski, Thm 3.6.1

is a little involved.

Now we can state and prove:

Thm 80: (Joyal extension theorem)

Let C be an ∞-category and $f : x \to y$ be a morphism in C. Then TFAE:



(i) f is an isomorphism.

(ii) Every morphism $a : \Lambda^n_0 \longrightarrow C$ with $a_{01} = f$ and $n \geq 2$ extends to Δ^n

(iii) Every morphism $b : \Lambda^n_n \longrightarrow C$ with $b_{n-1,n} = f$ and $n \geq 2$ extends to Δ^n.

proof: We prove (i) $\Leftrightarrow$ (ii) ((i) $\Leftrightarrow$ (iii) is dual).

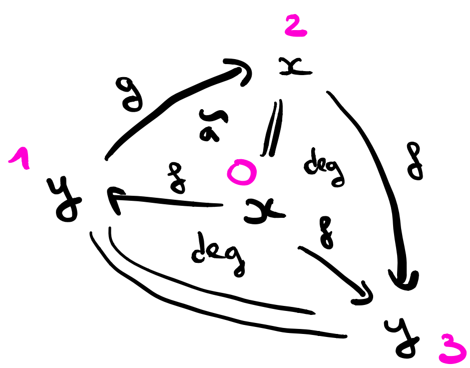
(ii) $\Rightarrow$ (i): This is the "easy" direction, and an argument we have already seen before when proving that Kan complexes are quasigroupoids.

- Take $n = 2$ and a given by $\begin{tikzcd} & y \\ x \arrow[ur, "f"] \arrow[rr, equal] & & x \end{tikzcd}$.

  By (ii), there exists an extension $\tilde{a} : \Delta^2 \to C$

  Put $g = \tilde{a}_{12}$. Then $[g][f] = \mathrm{id}_x$ in hC.

- Take $n = 3$ and $b : \Lambda^3_0 \longrightarrow C$ given by



By (ii), there is an extension $\tilde{b} : \Delta^3 \to C$ and $\tilde{b}_{123}$ proves that $[f][g] = \mathrm{id}_y$.

$\Rightarrow$ f iso with inverse g.

<u>(i) $\Rightarrow$ (ii):</u>

Let $a : \Lambda^n_0 \to C$ with $f := a_{01}$ isomorphism.

We have a pushout-join computation:

$$(\Delta^{\{0\}} \hookrightarrow \Delta^1) \boxed{\ast} (\partial\Delta^{\{2,\dots,n\}} \hookrightarrow \Delta^{\{2,\dots,n\}})$$

$$:= (\Delta^1 * \partial\Delta^{\{2,\dots,n\}}) \cup_{\Delta^{\{0\}} * \partial\Delta^{\{2,\dots,n\}}} (\Delta^{\{0\}} * \Delta^{\{2,\dots,n\}}) \hookrightarrow \Delta^1 * \Delta^{\{2,\dots,n\}}$$

$$= \left(\Delta^0 * \left(\bigcup_{i=2}^{n} \Delta^{\{2,\dots,n\} \setminus i} \right) \right) \cup_{\Lambda_0^{\{0,2,\dots,n\}}} \Delta^{\{0,2,\dots,n\}} \hookrightarrow \Delta^n$$

$$= \bigcup_{i=2}^{n} \Delta^{\{1,2,\dots,n\} \setminus i} \cup_{\Lambda_0^{\{0,2,\dots,n\}}} \Delta^{\{0,2,\dots,n\}} \hookrightarrow \Delta^n$$

$$= \Lambda^n_0 \hookrightarrow \Delta^n .$$

Using this and the join-slice adjunction, we get an equivalence of lifting problems

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r, hook] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r, "a"] \arrow[d, hook] & C \\
& \Delta^n \arrow[ur, dashed] &
\end{tikzcd}
\iff
\begin{tikzcd}
\Delta^{\{0\}} \arrow[r] \arrow[d, hook] & C_{/a|\Delta^{\{2,\dots,n\}}} \arrow[d, "q"] \\
\Delta^{\{0,1\}} \arrow[r, "g"] \arrow[ur, dashed] \arrow[dr, "f"'] & C_{/a|\partial\Delta^{\{2,\dots,n\}}} \arrow[d, "p"] \\
& C
\end{tikzcd}$$

C is an ∞-category and $\emptyset \subset \partial\Delta^{\{2,\dots,n\}} \subset \Delta^{\{2,\dots,n\}}$.

By Corollary 39, this implies that p and q

are right fibrations.

By Props and , p and q are thus conservative isofibrations.

Since δ is an iso and p is conservative, g is an iso.

Since q is an isofibration, there exists a lift.

This concludes the proof. $\square$

Rmk 81 This result can be interpreted as an homotopy coherence result; it tells you that an isomorphism, which is defined in terms of hC and so depends only on low-dimensional data, always comes with higher-dimensional coherence conditions.

This is remarkable, and forms the basis

of a lot of further developments of the theory of quasicategories.

The theorem admits a relative version.

Thm 82 (Joyal lifting theorem)

Let $p: C \longrightarrow D$ be an inner fibration between ∞-categories and $f \in C_1$ such that $p(f)$ is an isomorphism in D. TFAE:

(i) f is an isomorphism in C

(ii) For all $n \geq 2$, there is a lift in any diagram of the form:

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d] & C \arrow[d, "p"] \\
& \Delta^n \arrow[r] \arrow[ur, dashed] & D
\end{tikzcd}$$

(iii) For all $n \geq 2$, there is a lift in any diagram of the form:

$$\begin{tikzcd}
\Delta^{\{n-1,n\}} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^n_n \arrow[r] \arrow[d] & C \arrow[d, "p"] \\
& \Delta^n \arrow[r] \arrow[ur, dashed] & D
\end{tikzcd}$$

$\square$

We omit the proof, which is a variant of the previous one.

There are dual formulations of Joyal extension and liftings in terms of (co)slices.

Prop 83: Let C be an ∞-category.

(a) Let $f: x \to y$ be a morphism in C. TFAE:

(i) f is an isomorphism.

(ii) $C_{f/} \longrightarrow C_{x/}$ trivial fibration

(iii) $C_{/f} \longrightarrow C_{/y}$ trivial fibration.

(b) Let D be another ∞-cat. and $F : C \to D$ an inner fibration. Let $g : x \to y$ be a morphism in C such that $F(g)$ is an iso. TFAE

(i) g is an isomorphism.

(ii) $C_{g/} \longrightarrow C_{x/} \underset{D_{F(x)/}}{\times} D_{F(g)/}$ is a trivial fibration.

pullback-slices

(iii) $C_{/g} \longrightarrow C_{/y} \underset{D_{/F(y)}}{\times} D_{/F(g)}$ is a trivial fibration.

proof: We do a) (i) $\iff$ (ii). As before, the key is an equivalence of lifting problems: for $n \geq 0$, we have:

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & C_{g/} \arrow[d] \\ \Delta^n \arrow[r] & C_{x/} \end{tikzcd} \iff \begin{tikzcd} \Delta^1 * \emptyset \arrow[r] \arrow[rr, bend left, "g"] & (\Delta^1 * \partial\Delta^n) \cup (\{0\} * \Delta^n) \arrow[r] \arrow[d] & C \\ & \Delta^1 * \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

The RHS is

$$\begin{tikzcd}
\Delta^1 \arrow[r] \arrow[rr, bend left] & \Lambda^{1+1+n}_0 \arrow[r] \arrow[d] & C \\
& \Delta^{1+1+n} \arrow[ur, dashed] &
\end{tikzcd}$$

which is precisely the lifting problem in the Joyal extension theorem. So the Joyal extension theorem is in fact equivalent to a) (i) $\Leftrightarrow$ (ii) (together with a) (i) $\Leftrightarrow$ (iii))

$\square$

Rmk 84: The Joyal lifting theorem and its dual version motivate the definition of the last - but not least - major class of fibrations in the theory of quasicategories: Cartesian and Cocartesian fibrations.