$\underline{n=1}$: equivalent to part 1) of Def

$\underline{n=2}$ equivalent to surjectivity in part 2) of Def 9.

$\underline{n=3}$ equivalent to injectivity in part 2) of Def 9. $\square$

. The fibers of left/right fibrations are Kan complexes. This has some further consequences.

Recall that $\widetilde{sSet}$ is a simplicial category.

Let C be an ∞-category. The slice category $\widetilde{sSet}_{/C}$ has itself a structure of simplicial category.

Prop 12: Let $\begin{cases} p: D \to C \\ q: D' \to C \end{cases}$ be left fibrations.

between ∞-categories.

The simplicial set $\widetilde{sSet}_{/C}(p, q)$ of $\widetilde{sSet}_{/C}$ is a Kan complex.

Cor 13: Let C be an ∞-category. The simplicial subcategory $\widetilde{\mathrm{LFib}}(C)$ of $sSet_{/C}$ spanned by left fibrations is locally Kan.

Def 14: The ∞-category of left fibrations $\mathrm{LFib}(C)$ is defined as $N_\Delta(\widetilde{\mathrm{LFib}}(C))$.

- Let $\mathrm{Spc}_* := \mathrm{Spc}_{*/}$ be the ∞-category of pointed spaces. The canonical functor $\mathrm{Spc}_* \longrightarrow \mathrm{Spc}$ is a left fibration.

Rmk: I mistakenly claimed earlier that Spc_* is isomorphic to $N_\Delta(\widetilde{\mathrm{Kan}}_{\Delta^0/})$.

The situation is slightly more complicated:

there is a natural functor

$$N_\Delta(\widetilde{\mathrm{Kan}}_{\Delta^\circ/}) \longrightarrow N_\Delta(\widetilde{\mathrm{Kan}})_{\Delta^\circ/}$$

which is an equivalence of ∞-categories.

See Kerodon, Prop. 5.5.3.5.

- We can now state the main theorem of this section:

Thm 15 : (Joyal, Lurie)

There is an equivalence of ∞-categories

$$\int : \mathrm{Fun}(C, \mathrm{Spc}) \xrightarrow{\sim} \mathrm{LFib}(C)$$

which on objects sends a functor $F: C \longrightarrow \mathrm{Spc}$ to the pullback:

$$\begin{tikzcd}
\int F \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Spc_* \arrow[d] \\
C \arrow[r] & Spc
\end{tikzcd}$$

Dually, there is an equivalence of categories

$$\int : \mathrm{Fun}(C^{op}, Spc) \xrightarrow{\sim} \mathrm{RFib}(C)$$

which on objects sends $G : C^{op} \to Spc$ to the pullback:

$$\begin{tikzcd}
\int G \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Spc_*^{op} \arrow[d] \\
C \arrow[r, "G^{op}"] & Spc^{op}
\end{tikzcd}$$

We can thus say that $Spc_* \to Spc$ (resp. $Spc_*^{op} \to Spc^{*}$) is the universal left (resp. right) fibration.

Proofs:

- This theorem has a number of different proofs in the litterature; among them:

I) [HTT, Thm 2.2.1.2]

II) [Cisinski, Thm 5.4.5 and 7.8.9] ($\sim$)

III) Heuts and Moerdijk, <u>Left fibrations and homotopy colimits II</u> — the simplest?

(See also Beardsley-Wong, <u>Operadic nerve ...</u>, App B)

- All of them require presenting the equivalence using model structures. The proofs I) and III) also rely on the comparison with simplicial categories as model for ∞-categories.

- Let me just mention the necessary model structures (in Lurie and Heuts-Moerdijk).

+ Let $\mathcal{C}$ be a simplicial category. There is a simplicial category of simplicial functors $\mathrm{Fun}(\mathcal{C}, \widetilde{sSet})$, and we can equip it with a projective model structure (with respect to $sSet_{\text{Kan-Quillen}}$):

  * weak equivalences are objectwise weak homotopy equivalences.

  * fibrations are objectwise Kan fibrations.

This is a simplicial model structure, and if $\mathcal{C}$ is locally Kan the associated ∞-category is equivalent to $\mathrm{Fun}(N_\Delta(\mathcal{C}), \mathrm{Spc})$.

+ Let X be a simplicial set. Then

$s\widetilde{Set}_{/X}$ has a (simplicial) model structure, the covariant model structure, where

* the cofibrations are the monomorphisms
* the fibrant objects are the left fibrations

(Recall that by a result of Joyal this determines the model structure uniquely if it exists, see e.g. [Riehl, Categorical homotopy theory, Thm 15.3.1].)

One can describe the weak equivalences quite precisely, see [HTT, Definition 2.1.4.5].

If $X = C$ is an ∞-category, the associated ∞-category is equivalent to $LFib(C)$.

· The equivalence of the theorem is

obtained by "deriving" a Quillen equivalence between simplicial model categories. But what does this mean?

Rmk: Let $F : M \rightleftarrows N : G$ be a Quillen equivalence. It is natural to expect it to give rise to an equivalence of ∞-categories; this is true (Mazel-Gee, "Quillen adjunctions induce adjunctions of quasicategories"), but surprisingly subtle:

- In this course, we only discussed how to associate ∞-categories to simplicial model categories. It is possible more generally, but this requires other constructions.

- Even if M, N are simplicial, neither F nor G preserves bifibrant objects in general. (they each preserve half!)

Nevertheless, Lurie proves a result in HTT which is sufficient for the theorem above:

Thm : [HTT, Corollary A.3.1.12]

Let $\mathcal{M}$, $\mathcal{N}$ be simplicial model categories.

Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen equivalence.

Assume that : - Every object of $\mathcal{M}$ is cofibrant.

- G is a simplicial functor.

Then G induces a functor of simplicial categories $\mathcal{N}_{cf} \xrightarrow{G_{cf}} \mathcal{M}_{cf}$, and the h.c. nerve

$$N_\Delta(\mathcal{N}_{cf}) \xrightarrow{N_\Delta(G_{cf})} N_\Delta(\mathcal{M}_{cf})$$

is an equivalence of ∞-categories. $\square$

This is proven by checking that G_{cf} is a Dwyer-Kan equivalence of simplicial cats.

Rmk 16 : This formulation of the theorem is unsatisfying because it only describes what happens on objects. It would be better to have a purely ∞-categorical caracterisation of the functor, but I don't know of one!

Terminology:

Lurie calls this equivalence the "straightening / unstraightening" equivalence:

- unstraightening : $\mathrm{Fun}(C, \mathrm{Spc}) \longrightarrow \mathrm{LFib}(C)$
- straightening : $\mathrm{LFib}(C) \longrightarrow \mathrm{Fun}(C, \mathrm{Spc})$.

- Let us explore a little the "straightening" process. We start with a left fibration $p : D \longrightarrow C$. We won't quite produce a functor $C \to \mathrm{Spc}$, but a functor $hC \longrightarrow h\mathrm{Spc} \simeq h\mathrm{Kan}$.

- For $c \in C$, the fiber D_c is a Kan complex.

- Let $f : c \to c'$ be a morphism in C.

  The inclusion $D_c \simeq D_c \times \{0\} \hookrightarrow D_c \times \Delta^1$ is left anodyne because $\{0\} \hookrightarrow \Delta^1$ is. (Special case of Prop III.19)

So we can solve the lifting problem:

$$\begin{tikzcd}
D_c \arrow[r, hook] \arrow[d, hook] & D \arrow[d, "p"] \\
D_c \times \Delta^1 \arrow[r] \arrow[ur, dashed] & \Delta^1 \arrow[r, "\gamma"] & C
\end{tikzcd}$$

Restricting $\dashrightarrow$ to $D_c \times \{1\}$, we get

$$\gamma_! : D_c \longrightarrow D_{c'} .$$

$\gamma_!$ is not uniquely determined. However it is determined up to homotopy, and:

Lemma 17: The assignement

$$c \rightsquigarrow D_c \quad , \quad \gamma \rightsquigarrow [\gamma_!]$$

determines a functor $hC \longrightarrow h\,Kan$.

proof: Let $\sigma : \Delta^2 \longrightarrow C$ be a 2-simplex

$$\begin{tikzcd} & V \arrow[dr, "g"] & \\ U \arrow[ur, "f"] \arrow[rr, "h"'] & & W \end{tikzcd}$$

. Choose $f_!, g_!, h_!$

Using the fact that $D_c \times \{0\} \hookrightarrow D_c \times \Delta^2$ is also left anodyne, one shows that $[h_!] = [g_!] \circ [f_!]$ in $h\mathrm{Kan}$.

(see [HTT, Lemma 2.1.1.4]) $\square$

Rmk : The functor classified by a given left/right fibration is only defined up to isomorphism in $\mathrm{Fun}(\mathcal{C}^{(op)}, \mathrm{Spc})$, so "up to a contractible choice." This will apply to all the examples in the next section.

4) Applications

Terminology: As in the 1-categorical case, we call a functor $C^{op} \longrightarrow Spc$ a presheaf on the ∞-category C and sometimes write $PSh(C) := Fun(C^{op}, Spc)$.

- Representable and corepresentable functors

Let C be an ∞-category and $x \in C$.

We have:

- a left fibration $C_{x/} \longrightarrow C$
- a right fibration $C_{/x} \longrightarrow C$

The functors classified by them are:

- the corepresentable functor

$$R^{x} : C \longrightarrow Spc$$

- the representable functor (or presheaf)

$$R_x : C^{op} \longrightarrow Spc$$

Let us compute their values on objects:

$R^x(y)$ is by definition isomorphic in Spc to $(C_{x/})_y$ and $R_x(y)$ is iso. to $(C_{/x})_y$.

These simplicial sets are variants of the mapping spaces $map_C(x,y)$.

Def 18: Let C be an ∞-category and $x, y \in C_0$. The left(-pinched) mapping space $map^L_C(x,y)$ is defined as

$$map^L_C(x,y) := (C_{x/})_y .$$

More concretely, for any $n \geq 0$, we have

$$\mathrm{map}^L_C(x,y)_n = \left\{ \sigma \in C_{n+1} \;\middle|\; \begin{array}{l} \sigma(0)=x\,,\ d_0(\sigma) \\ \text{constant map } \Delta^n \to \{y\} \to C \end{array} \right\}$$

Dually, the right(-pinched) mapping space

$$\mathrm{map}^R_C(x,y) := \left(C_{/x}\right)_y$$

More concretely, for any $n \geq 0$, we have

$$\mathrm{map}^R_C(x,y)_n = \left\{ \sigma \in C_{n+1} \;\middle|\; \begin{array}{l} d_{n+1}(\sigma) \text{ constant map at } x \\ \sigma(n+1) = y \end{array} \right\}$$

$\square$

As fibers of left/right fibrations, those are Kan complexes. Moreover:

Prop 19 : There are natural monomorphisms

$$\mathrm{map}^L_C(x,y) \hookrightarrow \mathrm{map}_C(x,y) \hookleftarrow \mathrm{map}^R_C(x,y)$$

which are homotopy equivalences. $\square$

See [Kerodon, Prop 4.6.6.9].

- Because of this, it is also customary to write:

$$\begin{cases} h_x = \mathrm{map}_C(-, x) : C^{op} \longrightarrow \mathrm{Spc} \\ h^x = \mathrm{map}_C(x, -) : C \longrightarrow \mathrm{Spc} \end{cases}$$

- These new mapping spaces lead to reformulations of initial / terminal objects.

<u>Cor</u> : Let C be an ∞-category and $x \in C_0$. TFAE:

(i) x is terminal.

(ii) $\forall y \in C_0$, $\mathrm{map}^L_C(x, y)$ is contractible.

(iii) ——— map^R_C ———

(iv) ——— map_C ———

proof: By Prop , (ii) – (iv) are equivalent. We check (i) $\Leftrightarrow$ (iii).

- (i) is equivalent to: $C_{/x} \longrightarrow C$ is a trivial fibration.

- (iii) is equivalent to: $\forall y \in C_0$,

$$(C_{/x})_y = \mathrm{map}^R_C(y, x) \text{ is contractible.}$$

Since $C_{/x} \to C$ is a right fibration, these two conditions are eq. by Prop . $\square$

- There is a recognition principle for (co)representable functors as in 1-cat. theory:

Lemma 20: Let $F : C^{op} \longrightarrow \mathrm{Spc}$ be a functor. TFAE:

(i) F is isomorphic to a representable functor h_c in $\mathrm{Fun}(C^{op}, \mathrm{Spc})$.

(ii) $\mathfrak{H}F : \mathfrak{H}C^{op} \longrightarrow h\mathrm{Kan}$ is a representable functor of $\mathbb{R}$ Kan-enriched categories.

(ii) The ∞-category $\int F$ has a terminal object $\tilde{c}$.

In that situation, $p_F(\tilde{c}) = c$.

Proof: [HTT, 4.4.4.1 - 4.4.4.5]. $\square$

Mapping spaces

We have treated the functoriality of mapping spaces separately in each variable. We also expect a functor

$$\mathrm{Map}_C(-,-) : C^{op} \times C \longrightarrow \mathrm{Spc}$$

. For this, we should construct either a left or right fibration to $C^{op} \times C$ (since $(C^{op} \times C)^{op} \simeq C^{op} \times C$ canonically).

Def 21: Let C be an ∞-category. The twisted arrow ∞-category $\tilde{O}(C)$

is the simplicial set

$$\tilde{O}(C)_n := sSet(\Delta^{n,op} * \Delta^n, C)$$

There are morphisms

$$\tilde{O}(C) \xrightarrow{s} C^{op} \quad \text{and} \quad \tilde{O}(C) \xrightarrow{t} C$$

induced by the inclusions

$$\Delta^{\bullet,op} \hookrightarrow \Delta^{\bullet,op} * \Delta^{\bullet} \hookleftarrow \Delta^{\bullet}.$$

Rmk 22: This is a simplicial variant $\tilde{O}(C)$ of the twisted arrow category of a 1-category C, which is the category of elements of $Hom_C : C^{op} \times C \longrightarrow Set$.

$Ob\, \tilde{O}(C)$: morphisms in C

$Mor_{\tilde{O}(C)}(f, g)$: diagrams

$$\begin{tikzcd} x \arrow[d, "f"'] & z \arrow[l, "p"'] \arrow[d, "g"] \\ y \arrow[r, "q"'] & t \end{tikzcd}$$

Exercise: Let C be a 1-category.

Prove that $\tilde{\mathcal{O}}(N(C)) \simeq N(\tilde{\mathcal{O}}(C))$.

Prop 23: The morphism

$$(s,t) : \tilde{\mathcal{O}}(C) \longrightarrow C^{op} \times C$$

is a left fibration; in particular,

$\tilde{\mathcal{O}}(C)$ is an ∞-category.

proof: [Higher algebra, 5.2.1.3].

- There is another nice proof of this in Barwick-Glasman, <u>on the fibrewise effective Burnside ∞-category.</u>, § 1. $\square$

- We must check this has the correct fibers up to equivalence:

Lemma: Let C be an ∞-category, and $x \in \mathrm{Ob}(C)$. There is a canonical diagram

$$\begin{tikzcd} C_{/x} \arrow[r] \arrow[d] & \breve{\mathcal{O}}(C) \arrow[d] \\ \{x\} \times C \arrow[r] & C^{op} \times C \end{tikzcd}$$

in Cat_∞ which is homotopy-cartesian, i.e. the induced map

$$C_{/x} \longrightarrow \breve{\mathcal{O}}(C) \times_{(C^{op} \times C)} (\{x\} \times C)$$

is an equivalence of ∞-categories.

proof: This is contained in the proof of [Higher Algebra, Proposition 5.2.1.10].

5.2.1.11 $\square$

Def : By the Grothendieck construction, the left fibration $\widetilde{\mathcal{O}}(C) \longrightarrow C^{op} \times C$ classifies a functor:

$$C^{op} \times C \longrightarrow Spc$$

which by adjunction gives the Yoneda embedding:

$$y_C : C \longrightarrow PSh(C) := Fun(C^{op}, Spc)$$

Cor : Let $x \in C$. Then $y_C(x)$ is isomorphic to h_x in $PSh(C)$.

proof: This follows from Lemma and the Grothendieck construction.

Thm: (Yoneda lemma)

Let C be an ∞-category and $F \in PSh(C)$.

Let $x \in C_0$. There is a canonical iso.

$$\mathrm{map}_{PSh(C)}(h_x, F) \xrightarrow{\sim} F(x)$$

in Spc given by "evaluation at id_x".

proof: See eg [Cisinski, Theorem 5.8.13.(ii)]

Cor: (Yoneda embedding)

Let C be an ∞-category. The functor y_C is fully faithful; its essential image is spanned by representable presheaves.

proof: [Cisinski, Theorem 5.8.13.(i)] or [Higher Algebra, Proposition 5.2.1.11] combined with [HTT, Proposition 5.1.3.1].

Prop: The functor y_C preserves all (small) limits which exist in C.

proof: [HTT, Proposition 5.1.3.2].

Kan fibrations

Let $p: X \to C$ be a Kan fibration to an ∞-category. It is both a left and a right fibration, so one expects that there are two different, associated functors.

This is the case, and:

Prop: Let C be an ∞-category and $p: X \to C$ a functor. TFAE:

(i) p is a Kan fibration.

(ii) p is a left fibration, and for any $\gamma \in C_1$, $\gamma_!$ is an iso in $h\mathrm{Kan}$.

Equivalently, if F is the functor classified by p, then F factors through $\mathrm{Spc}^{\simeq}$.

(iii) dual condit° for right fibrations.

5) The $(\infty,1)$-Grothendieck construction

We need to identify the appropriate class of fibrations. Unfortunately this is more complicated than the definition of left/right fibrations.

Def: Let $p : X \to S$ be a morphism of simplicial set and $f : x \to y$ in X_1. We say that f is *p-cartesian* if one of the equivalent statements holds:

(i) For $n \geq 2$, any lifting problem

$$\begin{tikzcd}
\Delta^{\{n-1,n\}} \arrow[r] \arrow[rr, bend left, "f"] & \Lambda^n_n \arrow[r] & X \arrow[d, "p"] \\
& \Delta^n \arrow[u, hook] \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution.

(ii) The pullback-slice map

$$X_{/g} \longrightarrow X_{/y} \times_{S_{/p(y)}} Y_{/p(g)}$$

is a trivial fibration.

There is a dual notion of *p-cocartesian* morphism.

This should remind you of the Joyal lifting theorem. Indeed we can reformulate that theorem as

Prop: With the notations above, assume moreover that p is an inner fibration between ∞-categories. TFAE:

(i) g is an isomorphism.

(ii) f is p-cartesian and $p(f)$ is an iso.

(iii) f is p-cocartesian and $p(f)$ is an iso.

$\square$

Def: Let $p: X \to S$ be an inner fibration in sSet. Then p is a (co)cartesian fibration if every morphism in S admits a p-(co)cartesian lift: for every $g \in S_1$ there exists $f \in X_1$ and p-(co)cartesian with $p(f) = g$

Lemma: TFAE:

(i) p is a left fibration

(ii) p is a cocartesian fibration and every morphism in X is p-cocartesian.

Exercise: • Prove that (co)cartesian fibrations between ∞-categories are isofibrat°.

• Prove that: right fibrat° $\Leftrightarrow$ conservative cartesian fibrat°.

Rmk: Let $p : C \to D$ be a functor between 1-categories. The definition of p-cartesian morphisms in C is the same as Def . (ii) above, with "trivial fibration" replaced by "equivalence of category". One can then define the condition "p is a cartesian fibration". This is the notion relevant to the (1,1)-Grothendieck construction.

We then have:

$$p \text{ cartesian fibration} \\ \Leftrightarrow \\ N(p) \text{ cartesian fibration.}$$

One can define ∞-categories $\mathrm{Cart}(\mathcal{C})$ and $\mathrm{CoCart}(\mathcal{C})$ of cartesian / cocartesian fibrations over $\mathcal{C}$, and prove:

Thm (Lurie)

There are equivalences of ∞-categories

$$\begin{cases} \int : \mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty) \xrightarrow{\sim} \mathrm{Cocart}(\mathcal{C}) \\ \int : \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat}_\infty) \xrightarrow{\sim} \mathrm{Cart}(\mathcal{C}) \end{cases} \qquad \square$$

Rmk:

- It is possible to write down the universal (co)cartesian fibration and thus at least describe what $\int$ does on objects (pullback the universal fibrat°)
- Once again, the proof involves

deriving a Quillen equivalence of simplicial model categories. For this version of the theorem, as far as I know, there is only one reference:

[HTT, § 3.2].

Let's finish with two applications:

Ex: Let C be an ∞-category.

Consider $\mathcal{O}(C) := \mathrm{Fun}(\Delta^1, C) \xrightarrow{s} \mathrm{Fun}(\Delta^{\{0\}}, C) \simeq C$

$\mathcal{O}(C) \xrightarrow{t} \mathrm{Fun}(\Delta^{\{1\}}, C) \simeq C$

Then $\begin{cases} s : \mathcal{O}(C) \to C \text{ is a cartesian fibrat}^{\text{n}} \\ t : \mathcal{O}(C) \to C \text{ is a cocartesian fibration.} \end{cases}$

The functors classified by s, t can be written as

$C_{\bullet/} : C^{op} \longrightarrow \mathrm{Cat}_\infty$

$$C_{/\bullet} : C \longrightarrow Cat_\infty$$

and on objects and morphisms of C they recover (co)slices together with their functoriality..

Def : An adjunction between ∞-categories is a bicartesian (= cartesian + cocartesian) fibration.

$$a : X \longrightarrow \Delta^1 .$$ $\square$

By straightening in both directions, we get : $\begin{cases} \Delta^1 \longrightarrow Cat_\infty \\ (\Delta^1)^{op} \longrightarrow Cat_\infty \end{cases}$, that is

Functors $\begin{cases} f : X_0 \longrightarrow X_1 \\ g : X_1 \longrightarrow X_0 \end{cases}$

f is the left adjoint and g the right adjoint.

This definition is pretty strong and implies relatively easily other reasonable definitions of adjoints:

- natural equivalence

$$\mathrm{map}_{X_0}(-, g(-)) \simeq \mathrm{map}_{X_1}(f(-), -)$$

of functors to Spc.

- unit and counit + triangle identities

It is quite remarkable that all these definitions are actually equivalent!

" Adjunct° of ∞-categories are automatically homotopy coherent".

[Riehl-Verity, Homotopy coherent adjunct°(...)]

THE

END

?