# I) Simplicial Sets

Simplicial sets are combinatorial structures which occur in a few different ways in
- category theory
- algebraic topology
- abstract homotopy theory.

## 1) Definition

**def 1** The category Poset has

- $\mathrm{Obj}(\mathrm{Poset})$ = partially ordered sets
- $\mathrm{Mor}(\mathrm{Poset})$ = order-preserving maps.

$\square$

**def 2** The simplex category $\Delta$ is the full subcategory of Poset consisting of the objects consisting of the objects

$$[n] = \{0 < 1 < \dots < n\} \text{ for } n \in \mathbb{N}$$

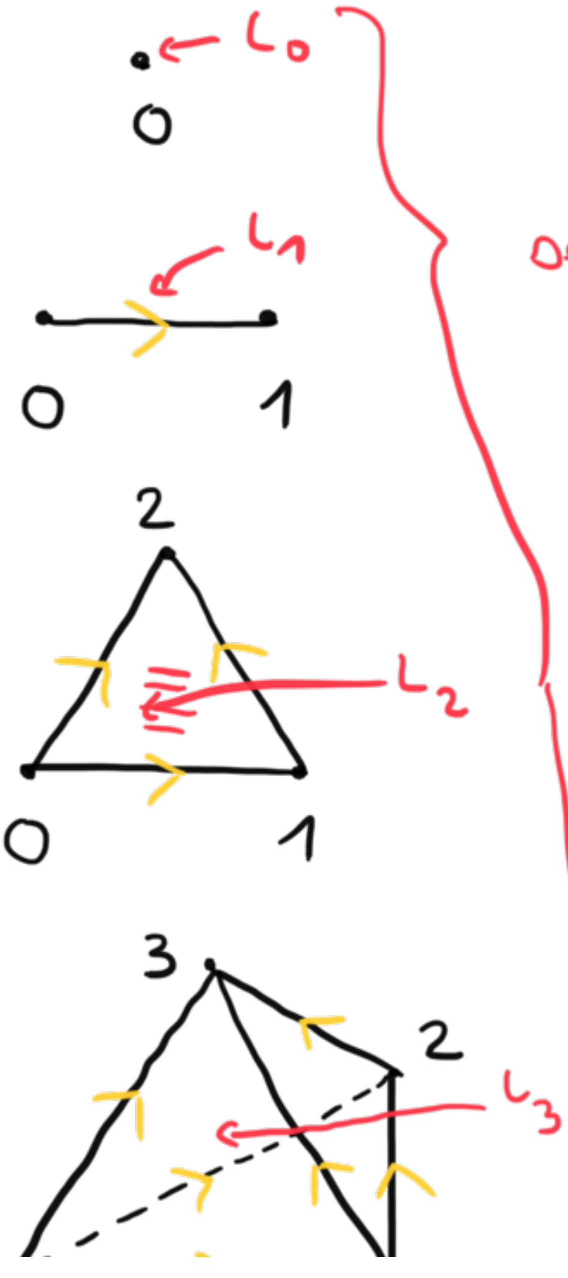
$\square$

## Why "simplex category"?

We can think of $[n]$ as indexing the vertices of the topological $n$-simplex

$$\Delta^n_{top} = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; \sum_{i=0}^{n} t_i = 1 \right\}$$

| $n$ | $\Delta^n_{top}$ + order |
| --- | --- |
| 0 | |
| 1 | |
| 2 | |
| 3 | |




see below

**Exercise** Show that $\Delta$ is equivalent to the full subcat of Poset consisting of all finite totally ordered sets.

**def 3** Let $C$ be any category.

- The category $sC$ of simplicial objects is the functor category:

$$sC := \mathrm{Fun}(\Delta^{op}, C)$$

- The category $cC$ of cosimplicial objects is the functor category

$$cC := \mathrm{Fun}(\Delta, C)$$

Explicitely:

$$\mathrm{Ob}(sC) : X : \Delta^{op} \longrightarrow C$$

$$\begin{cases} [n] \longmapsto X_n \in C \\ \delta : [n] \to [m] \longmapsto \delta^* : X_m \to X_n \end{cases}$$

$sC(X,Y)$ = nat. transformations $\alpha : X \to Y$, i.e.:

$\alpha_n : X_n \longrightarrow Y_n$ such that

$\forall \delta : [n] \to [m]$,
$$\begin{tikzcd} X_m \arrow[r, "\alpha_m"] \arrow[d, "\delta^*"'] \arrow[dr, phantom, "\equiv"] & Y_m \arrow[d] \\ X_n \arrow[r, "\alpha_n"] & Y_n \end{tikzcd}$$

We are particularly interested in

$$\boxed{\begin{aligned} sSet &:= \mathrm{Fun}(\Delta^{op}, Set). \\ &= PSh(\Delta) \end{aligned}}$$

"presheaves on $\Delta$"

Two entangled aspects

- categories of presheaves
- the structure of $\Delta$.

## 2) Presheaves

$C$ small category in this section.

**def 4** A functor

$$\begin{cases} F : C^{op} \longrightarrow \mathrm{Set} \\ G : C \longrightarrow \mathrm{Set} \end{cases}$$

is representable if

$\exists\, X \in C$ and a natural iso:

$$\begin{cases} F \simeq C(-, X) \\ G \simeq C(X, -) \end{cases}$$

$\square$

**Thm 5** (Yoneda Lemma)

$C$ (locally) category.

For any $\begin{cases} \text{functor } F : C^{op} \longrightarrow \text{Set} \\ X \in C \end{cases}$

there is a bijection:

$$\boxed{\begin{aligned} \mathrm{Nat}\big(C(-,X),F\big) &\cong F(X) \\ \alpha &\longmapsto \alpha_X(\mathrm{id}_X) \end{aligned}}$$

$\square$

**Cor 6** (Yoneda embedding)

The functor

$$\begin{aligned} y : C &\longrightarrow PSh(C) \\ X &\longmapsto C(-,X) \end{aligned}$$

is fully faithful. $\square$

**Thm 7:** (limits and colimits in PSh)

"(Co)limits in $PSh(C)$ are computed objectwise":

a) $PSh(C)$ is complete and cocomplete (i.e every diagram $F : I \to PSh(C)$ with $I$ small category has a (co)limit)

b) Let $F : I \longrightarrow PSh(C)$ be a diagram.

Then for all $X \in C$, we have

$$\big((\text{Co})\text{Lim}\, F\big)(X) \simeq (\text{Co})\text{Lim}\left(I \xrightarrow{F} PSh(C) \xrightarrow{ev_X} \text{Set}\right)$$

Proof: Let's do the case of colimits.

- Set is cocomplete so the RHS in b) is well-defined for any $X$.

- Choose a colimit cocone for every $X$:

$$\Big( F(i)(X) \longrightarrow \text{Colim}(ev_X \circ F) \Big)_{i \in I}$$

We want to show that [1)] assembles into a presheaf on $C$, and [2)] that one gets a colimit cocone for $F$.

<u>1)</u>: Let $f \in C(X, Y)$. We have

a natural transformation

$$f^* : ev_Y \circ F \Rightarrow ev_X \circ F$$

whose components are $F(i)(f)$.

By functoriality of colimits, there is an induced morphism between the cocones, and it gives the required presheaf "$\mathrm{Colim}\, F$", candidate for our colimit.

2) now follows from a computation, using only the def of morphisms of presheaves:

$$\mathrm{PSh}(C)\left(\text{"}\mathrm{Colim}\, F\text{"}, G\right)$$

$$\cong \left\{ \alpha_X : \mathrm{Colim}\left(ev_X \circ F\right) \to G(X) \middle| \dots \right\}$$

$$\cong \left\{ \alpha_{X,i} : F(i)(X) \to G(X) \middle| \dots \right\}$$

$$\cong \operatorname{Lim}_i \{ \alpha_{X,i} \mid \cdots \}$$

$$\cong \lim_i \mathrm{PSh}(C)(F(i), G) \qquad \square$$

**Rmk**

We choose a (co)limit for every $X$; this requires the axiom of choice. This can be avoided in some cases but in general we will use choice liberally.

**def 8:** Let $F \in \mathrm{PSh}(C)$.

The category of elements $\int F$ of $F$ has:

- Objects are pairs $(X, a)$ with $X \in C$, $a \in F(X)$
- Morphisms $(X, a) \to (Y, b)$

are morphisms $X \xrightarrow{f} Y$ in $C$

such that $F(f)(b) = a$.

There is a canonical forgetful functor $\Pi : \int F \longrightarrow C$

$$(X,a) \longmapsto X$$

$\square$

Rmk:

a) The functor $\Pi$ has a special property; it is a discrete right fibration

( $F : C \to D$ is a d.r.fib if for any $\begin{cases} d \xrightarrow{g} d' \text{ in } D \\ c' \in C \text{ with } F(c') = d' \end{cases}$,

$\exists!\ f : c \to c'$ with $F(f) = g$. )

b) In fact, any discrete right fibration $F : C \to D$ is equivalent to $\int G \longrightarrow D$ for $G \in PSh(D)$.

This is an instance of a general categorical pattern, the Grothendieck construction

$$\left\{ \begin{array}{l} \text{certain kinds} \\ \text{of functor} \\ \text{categories out} \\ \text{of } C^{(op)} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{certain kinds} \\ \text{of "fibrations"} \\ \text{to } C \end{array} \right\}$$

"lower categorical complexity"

which will be a major theme of the course.

**Thm 9:**

a) (Density of Yoneda embedding)

For any $F \in PSh(C)$, we have

$$\boxed{F \simeq \operatorname{Colim}\left(\int F \xrightarrow{\pi} C \xhookrightarrow{y} PSh(C)\right)}$$

i.e. $F$ is a colimit of representables in a canonical way.

b) (Free cocompletion property)

For any cocomplete category $D$, the functor induced by Yoneda:

$$\boxed{\mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(C), D) \xrightarrow{\sim} \mathrm{Fun}(C, D)}$$

is an equivalence.

colimit-preserving

Proof:

a) We construct a map $\longleftarrow$ ;

for $(X, a) \in \int F$, we have

$$a \overset{\text{Yoneda}}{\longleftrightarrow} \alpha : y(X) \longrightarrow F$$

which is exactly what we need to construct

$$\mathrm{Colim}\left(\int F \xrightarrow{y\pi} \mathrm{PSh}(C)\right) \longrightarrow F$$

To check it is an isomorphism, it is enough to do it after evaluation at $X \in C$.

- By Thm 7, we have

$$\operatorname{Colim}\left(\int F \xrightarrow{y \circ \pi} PSh(C)\right)(X)$$

$$\simeq \operatorname{Colim}\left(\int F \xrightarrow{\pi} C \xrightarrow{C(X,-)} Set\right).$$

and the map to $F(X)$ is given by $(Y, a) \in C \times F(Y)$,

$$f \in C(X,Y) \longmapsto F(f)(a).$$

We leave the proof this is a bijection as an exercise.

b) We construct a functor in the other direction:

$\mathrm{Fun}(C, D) \longrightarrow \mathrm{Fun}(\mathrm{PSh}(C), D)$

We start with $\begin{cases} G : C \to D. \\ F \in \mathrm{PSh}(C) \end{cases}$

a)

$$F = \mathrm{Colim}\left( \int F \xrightarrow{y \circ \pi} C \xhookrightarrow{y} \mathrm{PSh}(C) \right)$$

$$\tilde{G}(F) := \mathrm{Colim}\left( \int F \xrightarrow{\pi} C \xrightarrow{G} D \right)$$

- By using functoriality of colimits, can show that $\tilde{G}$ is a functor.

- To show that $\tilde{G}$ is colimit-preserving we show it has a right adjoint

$$H : D \longrightarrow \mathrm{PSh}(C)$$
$$d \longmapsto \left( c \longmapsto \underline{\underline{D(F(c), d)}} \right)$$

( Rmk if $F \dashv F'$, then

$H = y \circ F'$ by Yoneda embedding

$\rightsquigarrow$ In general $H$ is a

"formal" right adjoint of $F$.)

Let's check adjointness:

$$D(\tilde{G}(F), d)$$

$$\cong D\left(\mathrm{Colim}\left(\textstyle\int F \xrightarrow{G \circ \pi} D\right), d\right)$$

↑ def of $\tilde{G}$

$$\cong \mathrm{Lim}\left(\textstyle\int F \xrightarrow{G \circ \pi} D \xrightarrow{D(-,d)} \mathrm{Set}^{op}\right)$$

↑ representable functors send Colim to Lim.

. By Yoneda and def of $H$,

$$\begin{tikzcd} C \arrow[r, "G"] \arrow[d, "y"'] \arrow[dr, phantom, "\equiv"] & D \arrow[r, "{D(-,d)}"] & \mathrm{Set}^{op} \\ \mathrm{PSh}(C) \arrow[urr, "{\mathrm{PSh}(C)(-,H(d))}"'] & & \end{tikzcd}$$

Hence we have:

$$D(\tilde{G}(F), d)$$

$\mathrm{PSh}(C)(-, H(d))$

$$\simeq \mathrm{Lim}\left(\int F \longrightarrow C \xrightarrow{よ} PSh(C) \longrightarrow Set^{op}\right)$$

$$\simeq PSh(C)\big(F, H(d)\big)$$

reverse the arguments above! + a)

We now have functors in both direction and it is easy to them show it is an equivalence. $\square$

<u>Rmk</u> In the course of the proof, we see that

is a left adjoint

$$\mathrm{Fun}^{\mathrm{colim}}\big(PSh(C), D\big) \simeq \mathrm{Fun}^{L}\big(PSh(C), D\big)$$

i.e that any colimit-preserving functor out of $PSh(C)$ has a right adjoint, with a quite explicit formula.

There is a lot more to say about presheaves and we will come back to the subject.

We apply what we have learned to sSet.

Notat° For $X \in sSet$, we put $X_n := X([n])$ and call it the set of n-dimensional elements (or simplices) of $X$.

def 10 For $n \in \mathbb{N}$, the standard n-simplex $\Delta^n \in sSet$ is $y([n])$.

i.e. $(\Delta^n)_m = \left\{ \begin{array}{c} \text{order-preserving maps} \\ [m] \longrightarrow [n] \end{array} \right\}$

$\Delta^n$ has a universal n-dimensional element $\iota_n \in (\Delta^n)_n \; (\longleftrightarrow \mathrm{id}_{[n]})$.

$\square$

Cor 11: (Yoneda)

- For any $X \in sSet$ we have a natural bijection

$$X_n \cong sSet(\Delta^n, X)$$
$$f(\iota_n) \longleftarrow\!\shortmid f$$

- $$sSet(\Delta^n, \Delta^m) \cong \Delta([n],[m]) \cong (\Delta^m)_n$$

$\square$

Cor 12: ((co)limits in sSet)

They exist and are computed object-wise!

$\square$

Example

· For $X, Y \in sSet$, there is a cartesian product $X \times Y$ defined by $(X \times Y)_n := X_n \times Y_n$

Cor 13 (of Thm 9. a))

Every $X \in sSet$ satisfies

$$X \cong \underset{n,\, x \in X_n}{\operatorname{colim}} \Delta^n$$

i.e. is "glued" from standard simplices. $\square$

We also know how to define colim-preserving functors out of sSet, using Thm 9.b).

Example (geometric realisation)

. Start with

$$\Delta_{top} : \Delta \longrightarrow \mathrm{Top}$$
$$[n] \longmapsto \Delta^n_{top}$$

(for $f : [m] \longrightarrow [n]$, we put

$$\Delta_{top}(f) : \Delta^m_{top} \longrightarrow \Delta^n_{top}$$
$$(t_0, \dots, t_m) \longmapsto \Big( \sum_{i \in f^{-1}(0)} t_i \, , \, \dots \Big)\Big)$$

and apply Thm 9. b)

$\rightsquigarrow$ Geometric realisation

$$|-| : sSet \longrightarrow \mathrm{Top}$$

. Concretely,

$$|X| = \left( \coprod_{n \in \mathbb{N}} X_n \times \Delta^n_{top} \right) \Big/ \begin{array}{c} (f^*(x), t) \\ \sim \\ (x, \Delta_{top}(f)) \end{array}$$

(for all $f$ morphism in $\Delta$)

<u>Ex</u> $|\Delta^n| = \Delta^n_{top}$

⚠ This destroys the "order" information.

. $|-|$ has a right adjoint

$$\begin{aligned} S : \mathrm{Top} &\longrightarrow \mathrm{sSet} \\ X &\longmapsto \left([n] \mapsto \mathrm{Top}(\Delta^n_{top}, X)\right) \end{aligned}$$

the singular complex functor.

<u>Rmk</u> $\Delta^\bullet_{top} \in c\mathrm{Top}$

cosimplicial topological space.

More generally, if $Z^\bullet \in cC$,

$$\begin{aligned} C &\longrightarrow \mathrm{sSet} \\ c &\longmapsto \left([n] \mapsto C(Z^n, c)\right) \end{aligned}$$

The adjunction $|-| \dashv S$ is

The basic link between simplicial sets and homotopy theory: it is not an equivalence but induces: (Milnor)

$$\boxed{\quad sSet\left[\begin{matrix}\text{weak}^{-1}\\ \text{hom eq}\end{matrix}\right] \simeq Top\left[\begin{matrix}\text{weak}^{-1}\\ \text{hom eq}\end{matrix}\right]}$$