READING SEMINAR WS 18: CATEGORIES AND INFINITY-CATEGORIES

SIMON PEPIN LEHALLEUR

Introduction

Infinity category theory lies in the intersection of two major developments of 20th century mathematics: topology and category theory. Category theory is a very powerful framework to organize and unify mathematical theories. Infinity category theory extends this framework to settings where the morphisms between two objects form not a set but a topological space (or a related object like a chain complex). This situation arises naturally in homological algebra, algebraic topology and sheaf theory.

This reading seminar will recall the foundational ideas of usual category theory and then make the transition to homotopical algebra and infinity categories. By the end of the seminar, the student will be familiar enough with infinity categories that they can navigate texts written in this new language.

Guidelines for the talks

- The talks are given in English, should be given on the blackboard, and should last approximately 80 minutes to allow for 10 minutes of questions.
- Participants are expected to discuss their talk with me the week before they are scheduled to speak (and bring with them a draft of their talk notes). The default appointment for this discussion is Wednesday morning at 10am (the week before the talk) in room 108, Arnimallee 3 (if the participant is not available at this time, they should email before this date to arrange a different time).
- All required definitions and mathematical claims should be clearly stated; in particular, the definitions of all terms in italics in the descriptions below should be given.
- The speaker should make sure that the assumptions and the claim are clear to the audience, in order for the other participants to be able to follow proofs and explanations.

Program

0.1. **October 18th: Introduction (Simon Pepin Lehalleur).** Given by the lecturer.

0.2. **October 25th: Categories, Functors and natural transformations (Manuel Staiger).** Following [8, Chapter 1], introduce the basic definitions of category theory.
- Skip the historical introduction before 1.1.
- Present §1.1, with emphasis on some of the concrete examples of 1.1.3-4: Set, Top, Group, Ring, Mod_R, Man, Poset, Ch_R.
- Present the three exercises of §1.1.
- Explain categorical duality following §1.2.
- Do some of the exercises of §1.2, according to your personal preference.
- Explain the definition of a functor 1.3.1 and present some of the examples in 1.3.2: (i), (ii), (iii), (vi), (viii), (ix).
- Present the rest of §1.3,skipping 1.3.3-4 and 1.3.15.
- Explain the issue with duality of vector spaces at the beginning of §1.4.
- Explain the definition of a natural transformation between functors 1.4.1; the rest of the section consists of examples, explain some in 1.4.3.
- Give definitions 1.5.4, 1.5.7 and Theorem 1.5.9. Explain corollary 1.5.11.

0.3. **November 1st: Universal Properties, Representable Functors and the Yoneda Lemma (Alexander Feeß).** Following [8, Chapter 2], discuss how category theory gives a powerful formalisation of the idea of "universal properties".

- Define representable functors and explain many examples as in §2.1. State the questions at the end of §2.1 as motivation for the Yoneda lemma.
- State and prove the Yoneda lemma 2.2.4.
- Explain corollary 2.2.8.
- Present §2.3. The example 2.3.7 of tensor products, seen through the perspective of representable functors, is a very good one.
- Present the definition 2.4.1-2 and explain their relation with slice categories as in 2.4.6.

0.4. **November 11th: Limits and Colimits (Joshua Röher).** Following [8, Chapter 3], discuss the fundamental notion of limits and colimits of functors.

- Explain all of §3.1, perhaps skipping a few examples, but essentially everything here is important!
- Discuss how to construct all limits in the category of sets explicitely: 3.2.6 and 3.2.13.
- Explain what it means for a functor to preserve (co)limits 3.3.1, skip the rest of §3.3.
- Explain 3.4.2. State 3.4.6 and its dual 3.4.11 (without the reflection part). State Theorem 3.4.12.
- Give the definition of complete and cocomplete categories, and some examples and counterexamples as in the beginning of §3.5.

0.5. **November 15th: Adjoint Functors (Maria Delfin, Sarah Löser).** Following [8, Chapter 4], discuss the adjoint functors and its relationship to all the category theory we know so far.

- Present the definition of adjoint functors and some of the examples of §4.1.
- Explain units, counits and the alternative formulation of adjunction in §4.2.
- Skip §4.3.
- Present all the results in §4.4 and sketch at least one proof.
- Explain the fundamental results 4.5.1-3 and its applications 4.5.7-11. Explain Lemma 4.5.13.

0.6. **November 22th: Simplicial sets (Marie Brandenburg, Xiangying Chen).** A *simplicial set* is an abstract combinatorial object, which allows to model topological space up to homotopy. We have a set K_n for every $0 \leq n < \infty$, which we can think of as maps from an n-dimensional triangle into a space, and various morphisms $\delta_i : K_n \to K_{n-1}, \sigma_i : K_{n-1} \to K_n$ telling us how the triangles fit together.

- Define simplicial objects in a category as in [11, p. 8.1.4] and prove [11, pp. 8.1.1–3]. Explain example [11, p. 8.1.5]. Define morphisms of simplicial sets as natural transformations between functors and explain that simplicial sets form a category sSet.
- Define simplicial spheres and simplicial horns as in [9, pp. 5.1–3].
- Define the nerve $N(C)$ of a category C [3, Example 1.2]. Define the functor $N : \mathrm{Cat} \to \mathrm{sSet}$ and prove that it is fully faithful. Discuss the special case of the nerve of a partially ordered set, seen as a category.
- Define the geometric realisation of a simplicial set [11, p. 8.1.6]. Draw pictures of the geometric realisations of many of the examples considered so far.
- Define the singular simplicial set of a topological space [11, p. 8.2.4]. Prove that the geometric realisation is a left adjoint functor of the singular simplicial set functor $\mathrm{Sing}_\bullet(-)$.
- Define Kan simplicial sets [11, p. 8.2.7] and Kan fibrations [5, Ex.2.0.0.1].
- Explain the definition of simplicial homotopy groups of fibrant simplicial sets [11, p. 8.3.1]. State without proof that, for a Kan simplicial set $X_\bullet$, we have for all $n \geq 0$ and every $x_0 \in X_0$ that $\pi_n(X_\bullet, x_0) \simeq \pi_n(|X_\bullet|, x_0)$.
- Define the product of two simplicial sets $K_\bullet$ and $K'_\bullet$ by $(K \times K')_n = K_n \times K'_n$ and the obvious morphisms. State that it is the categorical product of $K_\bullet$ and $K'_\bullet$ in sSet.

- Given two simplicial sets K and K', define the mapping space $\operatorname{Map}(K, K')$, which is also a simplicial set, by $\operatorname{Map}(K, K')_n = \operatorname{Hom}(\Delta^n \times K, K')$. Prove that, for a fixed simplicial set K, the functor $\operatorname{Map}(K, -) : \operatorname{sSet} \to \operatorname{sSet}$ is a right adjoint of the functor $- \times K : \operatorname{sSet} \to \operatorname{sSet}$.

0.7. **November 29th: ∞-categories (Sagi Rotfogel, Konrad K. (?)** Infinity-categories, at least in the context of this seminar, will always be what is often called a *quasi-category*. A quasi-category is a special type of simplicial set, satisfying a lifting condition similar but weaker that the one of Kan simplicial sets discussed in the previous talk. We will see that one can in particular associate a quasi-category to any category, in a way that does not lose information. Quasi-categories are much more flexible, though: informally, they allow to have morphisms *between* morphisms (called 2-morphisms, and represented by 2-simplices), 3-morphisms between 2-morphisms, and so on.

- Define an ∞-category [3, Def.1.7], [5, Def.1.1.2.4]. Explain that, by definition, we have a (usual) category of ∞-categories which is a full subcategory of sSet.
- Define the opposite ∞-category of an ∞-category [5, §1.2.1].
- Show that for any category, its nerve is an ∞-category. Give an example of a category whose nerve is not a Kan complex. State and prove [5, Proposition 1.1.2.2] which caracterize those infinity categories which come from categories.
- It is difficult to write "by hand" many examples of infinity-categories which are neither nerves of categories nor Kan complexes. The simplest examples of the kind are ∞-categories which only have "non-trivial 2-categorical structure"; explain the case of the quasi-category of categories [**Rezk**] (you have to read [**Rezk**] to understand the construction).
- Define what it means for a functor between ∞-categories to be fully faithful and essentially surjective [5, p. 1.2.10.1]. Define subcategories of ∞-categories [5, §1.2.11].
- Define the space of right morphisms between objects in an ∞-category and explain that it is an ∞-category [5, Prop. 1.2.2.3], explain [5, Rmk. 1.2.2.4].
- Describe the homotopy category of an ∞-category, following [5, §1.2.3]; more precisely, just define $\pi(C)$ for C simplicial set and sketch the proof that it is a category [5, pp. 1.2.3.7–8], which we will denote hC. Note that passing to the homotopy category loses a lot of information; for instance, if K is a Kan simplicial set, then hK is equivalent to the discrete category on the set $\pi_0(K)$.
- This leads to the notion of isomorphism in an ∞-category C: two objects are said to be isomorphic iff they are isomorphic in the homotopy category hC (Lurie calls this equivalence, but many authors since have decided to call). State the caracterisation of isomorphism in terms of mapping spaces [5, Prop. 1.2.4.1].
- Define the simplicial set of functors between two ∞-categories [3, Def.2.1], [5, Not.1.2.7.2]. State that it is an ∞-category [3, Prop.2.5(i)], [5, p. 1.2.7.3].

0.8. **December 6th: Simplicial categories and ∞-categories (Amelie Flatt).** There is a more rigid model of $(\infty, 1)$-categories which is perhaps more intuitive and also useful to approach the more flexible model of Joyal and Lurie: simplicial categories, considered up to their natural homotopy equivalences. An important point is that certain simplicial categories gives rise to an ∞-category.

This lecture will cover:

- Explain the motivation for considering enriched categories as in the first paragraph of [10, Chapter 3].
- Briefly define symmetric monoidal categories as in [10, §3.1], and give the examples of categories with finite products (*cartesian* monoidal structures), hence Set, Top, and sSet, as well as the example of categories of modules over a ring. Note that, in this talk, Top should denote not the category of all topological spaces but a better behaved subcategory, the category of compactly generated weakly Hausdorff spaces, because it is cartesian closed (a notion which will be discussed below). This is really a minor technical issue which should be mentionned once and then studiously ignored.
- Define *enriched categories* as in [10, Def. 3.3.1]. Explain more concretely the examples of topological categories (categories enriched over Top), R-linear categories (categories enriched over $R - \operatorname{Mod}$ for R commutative ring) and in more detail simplicial categories

(categories enriched over sSet, the object of this talk). For this last one, follow the discussion in [10, §3.6]. Explain how to pass between topological and simplicial categories using the geometric realisation/singular set adjunction [5, end of p.18-top of p.19].

- Define *closed* symmetric monoidal categories [10, Def. 3.3.6], explain that such a category is enriched over itself via the internal homs, explain (using the construction of mapping spaces in the talk on simplicial sets) that sSet is a closed symmetric monoidal category. By the previous point, we also get a simplicial category structure on Top. A *cartesian closed* category is a category with finite products such that the cartesian monoidal structure is closed.
- Define equivalences of topological categories as in [5, Def. 1.1.3.6] and of simplicial categories as in [5, Def. 1.1.4.4].
- Define the simplicial nerve of a simplicial category and give some examples, following [5, pp. 1.1.5.1–9].
- State [5, Prop. 1.1.5.10, Cor.1.1.5.12]. Try to give an idea of the proof; for this, you need to understand how the inclusion of a simplicial cube minus a face into a simplicial cube can be "triangulated" and obtained by iteratively gluing maps of the form $\Lambda_i^n \subset \Delta^n$ (this is the meaning of "anodyne" in the proof of Prop. 1.1.5.10). It is not difficult to convince oneself of this fact using a picture; for a more rigourous proof, one can look at [5, pp. 2.1.2.6–8].
- As an application, define the ∞-category of *spaces* in the sense of [5, Def. 1.2.16.1], and also the similar ∞-category obtained from the simplicial category structure on Top.
- State [5, Thm. 1.1.5.13] and its equivalent formulation [5, Thm. 2.2.0.1], and explain following the end of [**h**]ow this can be interpreted as a comparison between the homotopy theory of simplicial categories and ∞-categories.

0.9. **December 13th: join and slices (Lucca Tiemens, Joaquim Boas).** The *join* of two topological spaces is the topological space we get by joining every point in one to every point in the other. This lecture shows how to define this for simplicial sets. A special case is when one space is a single point. This is called the *cone* for obvious reasons. Joins are needed for the definition of *slice* categories, which are needed for the definition of *limits* in infinity-categories. The "slice" of a morphism of topological spaces $f : X \to Y$ is something like the space of pairs (x, γ) where $x \in X$ is a point and $\gamma : [0, 1] \to Y$ is a path starting from $f(x)$.

This lecture will cover the following:

- Define the cone of a topological space, and draw the picture [4, pp.8-9].
- Define the join of two topological spaces, and draw the picture [4, p.9].
- Define the join of two simplicial sets [3, Def.2.11], [5, Def.1.2.8.1].
- Show that there are isomorphisms $\Delta^i \star \Delta^j \cong \Delta^{i+j+1}$.
- Define the right cone and left cone of a simplicial set, and describe them explicitly [3, Ex.2.14], [5, Not.1.2.8.4].
- Prove that for any two ∞-categories S, S', the join $S \star S'$ is an ∞-category [5, Prop.1.2.8.3].
- Define the overcategory $C_{/p}$ of a map p by its universal property [3, Prop.2.17], [5, Prop.1.2.9.2].
- Define $C_{/p}$ explicitly [5, Proof of Prop.1.2.9.2].
- State that $C_{/p}$ is an ∞-category, and invariant under categorical equivalences [5, Prop.1.2.9.3]. You can try to have a look at the proof, which is [5, Cor. 2.1.2.2], but it uses several notions which we don't have time to introduce.
- Define the undercategory by a universal property, and explicitly [5, Rem.1.2.9.5].
- Given a morphism of topological spaces $p : Y \to X$, explicitly describe the ∞-category $\mathrm{Sing}_\bullet(X)_{/\mathrm{Sing}_\bullet(p)}$.

0.10. **December 20th: Limits and Colimits in infinity categories (Viktor Tabakov, Alexei ?)** There is a theory of limits and colimits in infinity categories which generalizes the one for usual categories. This talk looks at the basic definitions, and also at examples in the infinity-category of topological spaces, which correspond to classical constructions of *homotopy limits and colimits*. The idea is that classical limits and colimits in topological spaces are not homotopy-invariant, and homotopy limits and colimits correct this.

- Define initial and final objects [3, §2.4], [5, §1.2.12.3]. State [5, Prop. 1.2.12.4, Cor. 1.2.12.5], prove the easy directions (not relying on results from [5, §2]). Explain in what sense initial/final objects are "unique" in an ∞-category [5, Prop. 1.2.12.9].
- Show that the ∞-category of a partially ordered set has an initial (resp. final) object if and only if it has a minimal (resp. maximal) element.
- Recall the definition of the ∞-category of topological spaces as the ∞-category associated to the simplicial category top. Give a more explicit description. For instance, using a cubical model for the singular complex construction, we see that the n-simplices consist of:
  (1) A set of $n+1$ topological spaces $X_0, \ldots, X_n$.
  (2) For each $i = 0, \ldots, n-1$ and $a = 1, \ldots, n-i$, a morphism $h_{i,i+a} : X_i \times \square_{top}^{a-1} \to X_{a+i}$.
  (3) The morphisms $h_{i,j}$ are required to satisfy the compatibility condition: For every a, b, the restriction of $h_{i,k+a+b}$ to $X_i \times \square_{top}^{a-1} \times \square_{top}^{b-1} \subseteq X_i \times \square_{top}^{a+b-1}$ is the composition $h_{i+a,i+a+b} \circ (h_{i,i+a} \times \mathrm{id}_{\square_{top}^{b-1}})$. Here, the inclusion $\square_{top}^{a-1} \times \square_{top}^{b-1} \subseteq \square_{top}^{a+b-1}$ is given by $((t_1, \ldots, t_{a-1}), (s_1, \ldots, s_{b-1})) \mapsto (t_1, \ldots, t_{a-1}, 0, s_1, \ldots, s_{b-1})$.
- Show that the one point topological space is a final object in the ∞-category of topological spaces. State that a topological space homotopy equivalent to a one point topological space is a final object.
- Define colimits and limits [5, Def.1.2.13.4].
- Observe that initial (resp. final) objects are colimits (resp. limits) of the empty diagram.
- Show that limits / colimits in (nerves of) partially ordered sets are infimums / supremums.
- Show that limits / colimits in (nerves of) categories are usual limits / colimits.
- Define pushout and pullback squares [5, Def.2.29].
- Define the *homotopy pushout* of a diagram $Z \xleftarrow{f} X \xrightarrow{g} Y$ of topological spaces as $\frac{Y \amalg [0,1] \times X \amalg Z}{(f(X) \sim \{0\} \times X, \ g(X) \sim \{1\} \times X)}$.
- Claim that this gives a pushout square in the ∞-category of topological spaces. (Optional) Show this claim. Explain that homotopy pushouts are homotopy invariant, in the sense that, for a map of diagrams as above whose components are weak equivalences, the resulting map is a weak equivalence.
- Define the *homotopy pullback* of a diagram $Z \xrightarrow{f} X \xleftarrow{g} Y$ of topological spaces as $\{(z, \gamma, y) \in Z \times \hom([0,1], X) \times Y : \gamma(0) = f(z), \gamma(q) = g(y)\}$. Claim that this gives a pullback square in the ∞-category of topological spaces.

0.11. **January 10th: Homotopical algebra and Model Categories (Lars Ran).** Model categories constitutes another framework which is very useful to study situations which go beyond usual category theory. For a long time, model categories were the best available substitute for ∞-categories; nowadays, they are still very useful for setting up the theory of ∞-categories and for carrying out computations.

- Explain the notion of localisation of a category at a set of maps, see for instance `https://ncatlab.org/nlab/show/localization`. The localisation always exists but is in general very difficult to understand. One of the goal of the theory of model categories is to get some control on the localisation in some situations.
- Present the definition of subcategories of weak equivalences, lifting properties and (functorial) weak factorisation systemes following [7, §14.1].
- Define model categories [7, Def. 14.2.1] and present the remarks and vocabulary given after the definition. Give [7, Def. 14.2.7] and the paragraph following it.
- State [7, Lemma 14.2.9]. It looks technical but is in fact a very useful statement throughout the theory. Its proof is a good example of how to use the axioms.
- Define the Quillen model structure on the category Top of topological spaces (as in previous talks, technically, one needs to restrict to so-called compactly generated topological spaces, but this is a minor detail and should not be insisted upon): see [7, Thm. 17.2.2] and the definition before.
- Describe the cofibrant objects of the Quillen model structure on Top [7, Rmk. 17.2.5], and explain in particular that CW-complexes are cofibrant. In particular, the existence of cofibrant replacements in this model structure is essentially a consequence of the classical fact that any topological space is weak homotopy equivalent to a CW-complex, and

the homotopy category of Top is equivalent to the category of CW-complexes, with mor-
phisms being homotopy equivalent classes of CW-maps. Mention that topologically trivial
morphisms, for instance covering spaces, are examples of Serre fibrations.

- Mention, without much details, that there is another model structure Top whose weak
  equivalences are homotopy equivalences [7, Def. 17.1.1].
- Define the projective model structure on the category $\mathbf{Cpl}(R - \mathrm{Mod})$ of unbounded chain
  complexes over a ring R [7, Def. 18.4.1, Thm. 18.4.2, Thm. 18.4.3]. Explain the relation-
  ship between cofibrant objects and projective R-modules [7, Prop. 18.5.2].
- Define cylinder objects and left homotopies as in [7, Def. 14.3.1] (Skip the good and very
  good variants). Explain that, by factoring the map $X \coprod X \to X$, there are always cylinder
  objects. Note that, in the Quillen model structure on Top (resp. the projective model
  structure on $\mathbf{Cpl}(R - \mathrm{Mod})$), there is an obvious choice of cylinder object on a space X
  (resp. a complex K), namely $X \times [0, 1]$ (resp. $K \otimes [\ldots \to 0R = R \to 0 \to \ldots]$), and that left
  homotopy then recovers the usual notions of homotopy in topology (resp. in homological
  algebra).
- State without proof or details that the notion of left homotopy for morphisms $f : A \to B$
  in a model category is well behaved (and in particular an equivalence relation) when the
  source A is cofibrant and the target B is fibrant. Define the homotopy category of a
  model category [7, Def. 14.4.5], noting that for any X, the object RQX is fibrant (by
  construction) but also cofibrant (because QX is cofibrant and the map $QX \to RQX$ is a
  cofibration).
- State [7, Prop. 14.4.6, Thm. 14.4.7].

0.12. **January 17th: The Quillen and Joyal model structures on Simplicial Sets (Daniel
K. (?))** This talk is devoted to two different model structures on the category sSet of simplicial
sets. The first one is the Quillen model structure, whose fibrant objects are Kan simplicial sets,
and which is Quillen equivalent to the Quillen model structure on Top. It plays a fundamental
role in many parts of homotopy theory, and leads in particular to the notion of simplicial model
categories. Simplicial model categories provide another source of quasi-categories, which is very
important in applications. The second one is the Joyal model structure, whose fibrant objects are
quasi-categories. Along the way, we will learn a little more about model categories.

- Define the cofibrations, fibrations and weak equivalences of the Quillen model structure
  on simplicial sets [7, Def. 17.5.1]. State that these form a model category. It would take
  too long to prove this result, but we could see some fragments of the proof which are of
  independent interest.
- The proof exploits in particular the fundamental relation between simplicial sets and topo-
  logical spaces. Recall the adjunction geometric realisation/singular complex construction
  between simplicial sets and (compactly generated weak Hausdorff, CGWH) topological
  spaces. In our reference [7, §17.5], the adjoint functors are denoted by $T = | - |$ and
  $S = \mathrm{Sing}$ but I suggest sticking to more standard notations. State [7, Lemmas 17.5.5-6]
  and explain the proofs (minus the technicalities with CGWH spaces; you can for instance
  prove the results only for finite CW-complexes).
- Explain [7, Lemma 17.6.1] which shows how to write any cofibration of simplicial sets as
  a kind of relative cell-complex (the class $\mathcal{I}$ of boundary inclusions is defined on p.360 and
  the relative cell-complexes are defined in [7, Def. 15.1.1]). State that there is a similar
  statement for trivial cofibrations, where the class $\mathcal{I}$ is replaced by the class $\mathcal{J}$ of horn
  inclusions, and one needs to consider *retracts* of relative $\mathcal{J}$-cell complexes.
- State and prove [7, Lemma 17.5.9]. State [7, Thm. 17.5.10, Cor.17.5.11] and sketch the
  proof of that corollary explained in [6, §16.2-3]. State the summary theorem [7, Thm.
  17.5.18], which collects the key features of the $(| - |, \mathrm{Sing})$ adjunction. In particular, at this
  point we know that the homotopy categories of topological spaces and simplicial sets are
  equivalent.
- (Omit in talk, have a look if you are interested) State [7, Lemma 17.6.2], explain that the
  map $j : Mi \to X \times I$ of the statement is a trivial cofibration and hence that lemma is a
  special case of the lifting properties predicted by the model category axioms, which turns

out to be a key case to complete the proof of the model category axioms. That's all we can say about the proof.

- Define Quillen adjunctions and Quillen equivalences [7, Def. 16.2.1]. For instance, the adjunction $(|-|, \mathrm{Sing})$ is a Quillen equivalence between the Quillen model structures on simplicial sets and topological spaces. Define left (and dually right) derived functors [7, Def. 16.1.1] and state [7, Prop. 16.2.2] (you can mention that the proof relies on the key Ken Brown lemma [7, Lemma 14.2.9] which was stated in the previous talk).
- Recall the adjunction between simplicial sets and simplicial categories [5, Def. 1.1.5.5], and the notion of equivalence of simplicial categories [5, Def. 1.1.4.4, Def. 1.1.3.6] from the talk on simplicial categories. Define the Joyal model structure, which is the model structure whose existence is claimed in [5, Thm. 2.2.5.1] (or rather, define the cofibrations, the weak equivalences, and claim that this provides a model structure). State that the fibrant objects of the Joyal model structure are exactly the ∞-categories [5, Thm. 2.4.6.1].
- Define simplicial model categories (see e.g. https://ncatlab.org/nlab/show/simplicial+model+category). Ex: sSet is a simplicial model category, but so is $\mathbf{Cpl}(R - \mathrm{Mod})$ and many other examples of interest. On the other hand, the Joyal model structure is not a simplicial model category, even though it is a model category structure on simplicial sets!
- Given a simplicial model category $\mathcal{M}$, we can associate an ∞-category as follows. First, prove that if X is a cofibrant object and Y is fibrant, then the corresponding mapping simplicial set $\mathrm{Map}_M(X, Y)$ (which I recall is defined as $(\mathrm{Hom}_M(X \times \Delta^n, Y))_{n \in \mathbb{N}}$) is a Kan complex; this is an exercise in understanding the definitions, please try to do it. This implies that the Joyal model structure on sSet is not simplicial, because there are ∞-categories X, Y (which are bifibrant in the Joyal model structure) such that the mapping space $\mathrm{Map}_M(X, Y)$ is not a Kan complex. Consider the subcategory of bifibrant objects $\mathcal{M}^{cf}$. The category $\mathcal{M}^{cf}$ inherits a structure of simplicial category from $\mathcal{M}$. Because of the previous result on mapping simplicial sets, we can apply [5, Prop. 1.1.5.10], which was discussed in a previous talk, to conclude that the simplicial nerve $N(\mathcal{M}^{cf})$ of the simplicial category M^{cf} is an ∞-category. This construction allows to connect the vast existing literature on model categories to the modern theory of infinity-categories.

0.13. **January 24th: Straightening and unstraightening, the Yoneda lemma for infinity categories (Gentaro M. (?))** Infinity-categories are complicated objects which encode a lot of data (morphisms, homotopies between morphisms, homotopies between homotopies, etc.) and constructing functors between ∞-categories seems like a very difficult combinatorial task. In particular, the analogue of presheaves in ∞-category theory are functors $C^{\mathrm{op}} \to \mathcal{S}$ with $\mathcal{S}$ is the ∞-category of spaces which was introduced before, and this seems hopeless. This talk will present some key ideas of Joyal and Lurie on how to make this possible; they generalize an important construction in usual category theory, the Grothendieck construction.

- Present [5, §2.1.1]. There is a dual notion of right fibration, present it in parallel. For another useful summary, look at [2, §1]. Define at the same time inner fibrations (as in [5, Definition 2.0.0.3].
- State [2, Thm 1.4]. Note that there Top is what we have called $\mathcal{S}$, and that a good name for the ∞-category $\mathrm{Fun}(C^{\mathrm{op}}, Top)$ is the ∞-category of presheaves (of spaces) on C. In [5], this theorem is a consequence of a Quillen equivalence of certain model categories [5, Thm 2.2.1.2], but the model-independent formulation is much easier to understand and remember, even though the proof seems to require to work with model categories.
- Explain the special cases of this theorem discussed in [2, Ex. 1.5-1.9]. For Ex. 1.8, you could have a look at [1, §1] to see what is involved in the proof.
- Present [2, §3] on (co)cartesian fibrations and its relation with the more general Grothendieck construction. Another good place to read about the Grothendieck construction is the beginning of [**Mazel-Grothendieck**].
- State [5, Cor. 5.1.2.3-4] which explain that ∞-categories of presheaves admit all limits and colimits and "how to compute them".
- Define the Yoneda embedding by adjunction from the construction in [2, Ex. 1.8]. State that it is fully faithful.

0.14. **January 31th: Adjoints in infinity categories, limits and colimits revisited (Florian M.).** We conclude this tour of general ∞-category theory by looking at the concept of adjoint functors. In Lurie's perspective, this is obtained as a special case of the theory of Cartesian fibrations from the previous talk. There is also an equivalent, perhaps more intuitive definition.

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0.15. **February 7th Stable infinity-categories (Louis Martini).** Stable ∞-categories are defined by

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0.16. **February 14th: Monoidal infinity-categories (Youshua Kesting).**

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