Models of curves and abelian varieties

I Introduction

1) Generalities

R Dedekind ring (normal noetherian integral domain of dimension ≤ 1)

$$K = \mathrm{Frac}(R)$$

$$\eta = \mathrm{Spec}(K) \longrightarrow S = \mathrm{Spec}(R) \longleftarrow s \quad \text{closed point } (\neq \eta)$$

ex
- fields
- discrete valuation rings: $\mathbb{Z}_p$, $k[[t]]$, ...
- rings of integers of a number field; $\mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{Z}\left[\frac{1+\sqrt{2}}{2}\right]$, ...
- rings of functions on a regular affine curve: $k[t]$, ...
- ...

def | X scheme over η. A model of X is a pair $(\mathfrak{X}, i)$ with $\mathfrak{X}$ S-scheme and $i : \mathfrak{X} \times_S \eta \xrightarrow{\sim} X$.
A morphism of models $(\mathfrak{X}, i) \longrightarrow (\mathfrak{X}', i')$ is

$$\begin{tikzcd} \mathfrak{X} \arrow[rr] \arrow[dr] & \equiv & \mathfrak{X}' \arrow[dl] \\ & S & \end{tikzcd} \quad \text{with} \quad \begin{tikzcd} \mathfrak{X} \times_S \eta \arrow[rr] \arrow[dr, "i"'] & \equiv & \mathfrak{X}' \times_S \eta \arrow[dl, "i'"] \\ & X & \end{tikzcd}$$

rmk The morphism i is often uniquely determined and we often omit it from the notation.

Goal: Given a class of varieties, want to understand models with specified properties.

ex: Can ask for $\mathfrak{X}/S$
- faithfully flat (almost always)
- of finite type over S (if X is)
- reduced, integral, normal, regular (if X is)
- separated, proper (if X is)
- smooth (if X is)
- group scheme (extending a group structure on X)

...

rmk : Let $X \hookrightarrow \mathbb{P}^n_\eta$ be projective, defined by homogeneous equations $F_1, \dots, F_m$. We get a projective flat model of X by taking the Zariski closure of X in $\mathbb{P}^n_S$

$$\Updownarrow$$

"chasing denominators in $F_1, \dots, F_m$"

Naive construction, provides raw material for more sophisticated constructions.

def : (for this intro)
A curve over η is a smooth projective geometrically connected scheme over η, of dimension ≤ 1.

def : An abelian variety over η is a smooth projective (geometrically) connected group scheme over η. ($\Rightarrow$ commutative !)

common ground: elliptic curves $\begin{cases} \text{curves of genus } 1 \text{ with } E(K) \neq \emptyset \\ \text{abelian varieties of dim. } 1 \end{cases}$

2) Four fundamental results

- Existence of minimal regular models of curves.
- Existence of Néron models of abelian varieties
- Stable reduction theorem for curves.
- Semi-abelian reduction theorem for abelian varieties.

Minimal regular models

The ideal model of a curve C would be a smooth projective one. When this exists, we say that C has <u>good reduction</u>.

If $g(C) \geq 1$, the model is then unique up to unique iso.

thm | (Lipman '78; Lichtenbaum-Shafarevich ~'66-'68)
Let C be a curve over η. Then C has projective regular models. Moreover, if $g(C) \geq 1$:
i) there exists one such model $\mathcal{C}^{reg}$ which is minimal: for any such model $\mathcal{C}$, any birational map $\mathcal{C} \dashrightarrow \mathcal{C}^{reg}$ is a morphism.
ii) there exists one such model $\mathcal{C}^{nc}$ whose reduced special fibers are normal crossings divisors, and minimal for this property.

- In particular, $\mathcal{C}^{reg}$ (resp. $\mathcal{C}^{nc}$) is a terminal object in the category of projective regular models of C (resp ...), hence unique up to a unique iso; C has good red $\iff \mathcal{C}^{reg}$ smooth
$\iff \mathcal{C}^{nc}$ smooth

Néron models

The ideal model of an abelian variety would be an <u>abelian scheme</u>, i.e, a smooth projective group scheme over S with connected fibers. When this exists, we say that the abelian variety has <u>good reduction</u>, and the abelian scheme model is unique up to a unique iso.

def | Let X/η be a smooth scheme. A <u>Néron model</u> of X is a smooth model $\mathcal{N}$ of X such that, for all smooth S-schemes $\mathcal{Z}$, the restriction map

$$\mathrm{Hom}_S(\mathcal{Z}, \mathcal{N}) \longrightarrow \mathrm{Hom}_\eta(\mathcal{Z}_\eta, X)$$

is a bijection.

- In particular, N is terminal in the category of smooth models of X.
- If X is a group scheme over η, then N is a group scheme over S in a compatible way.

<u>thm</u> (Néron '64)
Let A be an abelian variety over η. Then A admits a Néron model, which is quasi-projective.

- The structure of the $\begin{cases} \text{minimal regular model of a curve} \\ \text{Néron model of an abelian variety} \end{cases}$

can be quite complicated. The other two major results tell us that, if we are ready to extend K, the situation improves a lot.

Stable reduction

<u>def</u>
- Let k be an algebraically closed field.
- A <u>semi-stable curve</u> over k is a reduced finite type k-scheme C of dimension 1, such that C has only nodal singularities: if $x \in C$ is singular, then $\widehat{\mathcal{O}}_{C,x} \simeq k[[u,v]]/(uv)$.
- We say that C is <u>stable</u> if C is moreover proper, connected, of arithmetic genus ≥ 2, and if any irred. component $\simeq \mathbb{P}^1$ meets other components in ≥ 3 pts.
- A morphism $\mathcal{C} \to T$ is a <u>stable</u> (resp. <u>semistable</u>) <u>curve</u> if it is proper flat and its geometric fibers are stable (resp. semistable) curves.

<u>rmk</u> Stable curves have other equivalent, more conceptual definitions.

rmk The total space of a (semi-)stable curve over a regular base like S may not be regular but tends to have "mild" singularities. (basis of de Jong's res. of sing by alterations!)

thm (Stable reduction; Deligne-Mumford '69)
Let C be a curve over η, of genus ≥ 2.
There exists a finite separable extension L/K such that C_L admits a stable model over the normal closure S_L of S in L.

· The stable model is unique if it exists, while there can be many different semistable models. In fact:

prop C admits a stable model
$\Updownarrow$
$\mathscr{C}^{reg}$ semistable
$\Updownarrow$
$\mathscr{C}^{nc}$ semistable.

Semi-abelian reduction

def A semiabelian variety over a field k is a smooth k-group scheme G which is an extension of an abelian variety by a torus:

$$1 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 1$$

($\Rightarrow$ commutative)

Recall also that a finite type group scheme G over a field has an identity component G° (connected comp. of e_G)

thm (Semiabelian reduction; Grothendieck '67 - '68)
Let A be an abelian variety over η.
There exists a finite separable extension L/K such that, if $\mathcal{N}$ is the Néron model of A_L (over S_L), then for every $\sigma \in S$, $\mathcal{N}_\sigma^\circ$ is a semiabelian variety.

- Note that $\mathcal{N}_\sigma \neq \mathcal{N}_\sigma^\circ$ even when A already has semiabelian reduction.

rmk: This course will focus on the geometry of models, rather than on their numerous applications to arithmetic geometry, diophantine geometry and moduli theory.
Any serious treatment of any of these applications would require another course and another lecturer!
Nevertheless you are encouraged to force me to learn some of these and explain them to you.

II Weierstrass models of elliptic curves

- We start with elliptic curves. For this special case, there is a theory of models which is more elementary than both minimal regular models and Néron models

1) Weierstrass equations

def 1 Let R be a ring. A Weierstrass equation with coefficients in R is an homogeneous equation of the form

$$y^2 z + a_1 x y z + a_3 y z^2 = x^3 + a_2 x^2 z + a_4 x z^2 + a_6 z^3$$

- Our goal is to derive the main properties of such equations with as little computations as possible, and then forget about the equations!

(not recommended if you actually want to compute things)

Lemma 2 Let k be a field, $W \subseteq \mathbb{P}_k$ given by a Weierstrass equation. Then W is geometrically integral, smooth at $[0:1:0]$.

proof: – Writing the equation as $F(x,y,z) = 0$, we have

$$\frac{\partial F}{\partial z} = y^2 + a_1 x y + 2 a_3 y z - a_2 x^2 - 2 a_4 x z - 3 a_6 z^2$$

Hence $\frac{\partial F}{\partial z}([0:1:0]) = 1 \neq 0 \Rightarrow W$ is smooth at $[0:1:0]$.

- $W \cap V(z) = \{[0:1:0]\}$. Since $V(z)$ intersects all irreducible components of W (2 curves in $\mathbb{P}^2_k$ intersect!), and $[0:1:0]$ is a smooth point, we get W irreducible.
- W irreducible cubic curve. If W non-reduced, then necessarily $F = L^3$, and W everywhere non-reduced $\lightning$
- This argument applies over any field extension $\Rightarrow W$ geometrically integral. $\square$

Prop 3 Let (E, e) be an elliptic curve over a field k. There exists $x, y \in K(E)$ such that the birational map $\phi : E \longrightarrow \mathbb{P}^2_k, [x : y : 1]$ is a closed immersion whose image is cut out by a Weierstrass equation, and $\phi(e) = [0:1:0]$ unique pt in the line at ∞

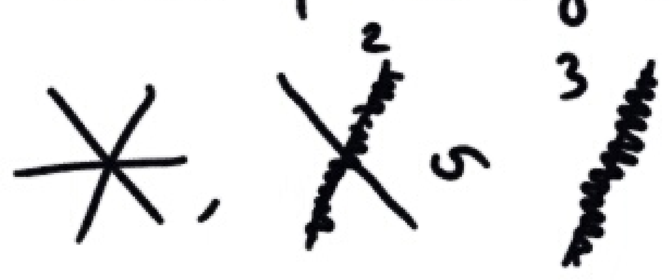
proof:
- RR $\Rightarrow \forall n \geq 1, \dim H^0(E, \mathcal{O}(ne)) = n$.
- Choose meromorphic functions x, y such that
  $\{1, x\}$ basis of $H^0(E, \mathcal{O}(2e))$
  $\{1, x, y\}$ basis of $H^0(E, \mathcal{O}(3e))$
- Then $\{1, x, y, x^2, xy, y^2, x^3\} \subseteq H^0(E, \mathcal{O}(6e))$

(Recall that if $k = \mathbb{C}$, $E^{an} = \mathbb{C}/\Lambda$ then can choose
$x = \wp$, $y = \wp'$ with $\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$ Weierstrass elliptic function.)

$\Longrightarrow \exists A_1, \dots, A_7 \in k$, not all 0,
$A_1 + A_2 x + A_3 y + \dots + A_7 x^3 = 0$

- By considering pole order at e and using $\dim H^0(E, \mathcal{O}(5e)) = 5$ see that $A_6 \neq 0$ and $A_7 \neq 0$.
- Subsitute $x \rightsquigarrow -A_6 A_7 x$, $y \rightsquigarrow A_6 A_7^2 y$ $\Longrightarrow$ (Weierstrass equat°)$\cdot A_6^3 A_7^4$
- Get $\phi : E \dashrightarrow \mathbb{P}^2$. Since E is smooth of dim ≤ 1 and $\mathbb{P}^2$ is proper, ϕ is automatically a morphism. Then $\mathrm{Im}(\phi)$ is given by Weierstrass equation.
- By composing with proj to $\mathbb{P}^1$ via $[x:1], [y:1]$, we see that $\deg(\phi)$ divides 2 and $3 \Rightarrow \deg(\phi) = 1$.
- To conclude, it remains to show that $\mathrm{Im}(\phi)$ is smooth. By Lemma 2, $\mathrm{Im}(\phi)$ is an integral cubic curve in $\mathbb{P}^2$.

• Assume that $\mathrm{Im}(\phi)$ is singular and let p be a singular point. Then p is of multiplicity exactly 2 (otherwise, $\mathrm{Im}(\phi)$ would be either



, hence not integral). This implies that the projection map $\mathbb{P}^2 \setminus \{p\} \longrightarrow \mathbb{P}^1$ restricts to a degree 1 rational map

$\mathrm{Im}(\phi) \dashrightarrow \mathbb{P}^1$, which composed with $E \to \mathrm{Im}(\phi)$

gives $E \dashrightarrow \mathbb{P}^1$ of degree 1 $\lightning$ $\square$

rmk: If $\mathrm{char}(k) \neq 2, 3$, there are simplified forms of Weierstrass equations; since we try to develop a purely geometric theory this is not so important for us.

prop 4: Let $W \subseteq \mathbb{P}^2_k$ be defined by a Weierstrass equation. Then W smooth $\Leftrightarrow$ $(W, [0:1:0])$ elliptic curve.

proof:

$\Leftarrow$ follows from prop 3.

$\Rightarrow$ follows from Hurwitz's genus formula and $[0:1:0] \in W(k) \neq \emptyset$.

Alternative argument for $\Rightarrow$: from Weierstrass equation, can write down

$$\omega = \frac{dx}{2y + a_1 x + a_3} = \frac{dy}{3x^2 + 2a_2 x + a_4} \in \Omega^1_{K(W)/k}$$

and show (assuming W smooth) that $\omega \in H^0(W, \Omega^1_{W/k})$ and that ω is everywhere non-vanishing.

A curve with such a 1-form has genus 1 $\square$

Here are some further geometric properties of Weierstrass equations which are not strictly necessary for what follows and which we state without proofs.

<u>def 5</u> · The <u>discriminant</u> Δ of a Weierstrass equation is
$$\Delta = 2^{-4} \operatorname{disc}_X\left(4(X^3 + a_2 X^2 + a_4 X + a_6) + (a_1 X + a_3)^2 \right) \in k.$$
· We also define
$$c_4 = (a_1^2 + 4a_2)^2 - 24(2a_4 + a_1 a_3) \in k$$

<u>prop 6</u> Any two Weierstrass equation for the same curve are related by a change of variables of the form $\begin{cases} x = u^2 x' + r \\ y = u^3 y' + s u^2 x' + t \end{cases}$ with $u \in k^\times$, $r, s, t \in k$.

<u>prop 7</u> If $\mathrm{char}(k) \neq 2,3$, using such a change of variable, we can put the equation in the form: $y^2 = x^3 + Ax + B$
and then $\begin{cases} \Delta = -16(4A^3 + 27B^2) \\ c_4 = AB. \end{cases}$

<u>prop 8</u> Let k be a field and W given by a Weierstrass equation.
(i) W non-singular $\Leftrightarrow \Delta \neq 0$.
(ii) If $\Delta = 0$, $c_4 \neq 0$, then W has a unique geometric singularity which is a node defined over k.
(note that the branches at the node may not be defined over k.)
(iii) If $\Delta = 0$, $c_4 = 0$, then W has a unique geometric singularity which is a cusp. It is defined over k unless $\mathrm{char}(k) \in \{2,3\}$ and k is imperfect.

· All this can be found in [Silverman, chap III] except the discussion of non-rational cusps on non-perfect fields which is nicely explained in [Conrad_models, p.15].