Quadratic enumerative geometry & the Deligne-Milnor formula

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I) Quadratic enumerative geometry

(also known as refined enumerative geometry
$\mathbb{A}^1$ - enumerative geometry)

Classical enumerative geometry

- Numerical invariants attached to algebro-geometric situations, with ~ 2 origins:
  - topological / motivic (eg intersection numbers)
  - coherent / K-theoretic (eg chern numbers)
- Invariants live in

$$CH_0(k) \simeq \mathbb{Z} \simeq K_0(k)$$

- Some of the most interesting formulas in enumerative geometry relate the two sides.

Ex: « Gauss - Bonnet in AG »

X smooth projective $/ k$

$$\chi(X) = \Delta_X \cdot \Delta_X \qquad \left(\begin{array}{l}\text{Lefschetz trace formula} \\ \text{for } \mathrm{id}_X\end{array}\right)$$

$$= (\pi_X)_*(e(T_X)) \qquad \left(\begin{array}{l}\text{self-intersection} \\ \text{formula for} \\ \Delta_X\end{array}\right)$$

$$= \sum_{p,q=0}^{n} (-1)^{p+q} \dim H^p(X, \Omega^q_X) \qquad \left(\begin{array}{l}\text{Hirzebruch-} \\ \text{Riemann-Roch} \\ \text{for } \pi_X\end{array}\right)$$

$$=: \chi_{coh}(X)$$

Quadratic refinements

- In some situations, can refine numerical invariants to the Grothendieck-Witt ring

$$\widetilde{CH}_0(k) \simeq GW(k) \simeq KH_0(k)$$

- $k^\times = (k^\times)^2 \implies GW(k) \simeq \mathbb{Z}$

  $\implies$ This is an arithmetic theory.

  (but $\mathbb{C}(X)$ is not quad. closed...)

- $k = \mathbb{R}$, $GW(\mathbb{R}) \xrightarrow[\sim]{(rnk, sgn)} \{(a,b) \in \mathbb{Z}^2 \mid a \equiv_2 b\}$

$$\begin{cases} \text{The rank recovers enumerative invariants} / \mathbb{C} \\ \text{The signature recovers real enumerative invariants.} \end{cases}$$

- Where do the refinements come from??

  On coherent side, Grothendieck duality is a source of non-degenerate symmetric bilinear forms.

<u>**Ex**</u> **Refining χ_{coh}**

X/k smooth projective of dimension n.

$$Hdg(X/k) := \left(\bigoplus_{i,j=0}^{n} H^i(X, \Omega^j)[j-i] \,,\, d = 0 \right)$$

is a perfect complex of k-vs, equipped with a non-degenerate symmetric bilinear form via

$$H^i(X,\Omega^j) \times H^{n-i}(X,\Omega^{n-j}) \xrightarrow{\text{cup}} H^n(X,\Omega^n) \xrightarrow{\mathrm{Tr}} k$$

$$\chi^{GW}_{coh}(X) := \left(H_{dg}(X/k), \mathrm{Tr} \right) \in GW(k)$$

. This is simpler than it looks!

$$\begin{cases} * \ n \text{ odd} \Rightarrow \chi^{GW}_{coh}(X) = \left(\dfrac{\chi_{coh}(X)}{2}\right) \cdot h \\ \qquad \text{is hyperbolic.} \\ * \ n \text{ even} \Rightarrow \chi^{GW}_{coh}(X) = \underbrace{\left(H^{\frac{n}{2}}(X,\Omega^{\frac{n}{2}}), \mathrm{Tr}\right)}_{\text{only non-hyperbolic contribution.}} + m \cdot h \end{cases}$$

$$\cdot\ k = \mathbb{R} \Rightarrow \begin{cases} \operatorname{rnk}\left(\chi^{GW}_{coh}(X)\right) = \chi(X_{\mathbb{C}}) \\ \operatorname{sgn}\left(\chi^{GW}_{coh}(X)\right) = \chi_{top}(X(\mathbb{R})) \end{cases}$$

Abelson

Quadratic refinements on the motivic side come from motivic homotopy theory.

Motivic homotopy theory

(also known as $\mathbb{A}^1$-homotopy theory)

- Morel - Voevodsky have graciously provided us with a common framework for many cohomology theories in algebraic geometry over a base scheme B, the stable motivic homotopy category $SH(B)$.

- The theory is modelled on stable homotopy theory of topological spaces and the category SH of spectra:

$$\begin{cases} SH = Top_*[(S^1)^{\wedge -1}] \longleftarrow \text{spectra} \\ SH(B) = L_{\mathbb{A}^1, Nis} P(Sm_B, Top_*)\,[(\mathbb{P}^1_\infty)^{\wedge -1}] \end{cases}$$

- $SH(B)$ is a tensor triangulated category.

  (symmetric monoidal stable ∞-category)

- Mixture of algebraic geometry and topology:

$$\Sigma^\infty : Sm_{B,*} \longrightarrow SH(B) \longleftarrow SH : \underset{\text{presheaf}}{cst}$$

Ex i) Spheres

$$\mathbb{S}^{p,q} := cst(S^{(p-q)}) \wedge (\Sigma^\infty(\mathbb{G}_m, 1))^{\wedge q}$$

(q records the Tate twist)

Using $\mathbb{S}^{p,q}$, can define (bigraded) stable motivic

Homotopy groups for any $E \in SH(B)$.

ii) Thom spaces $\quad V \longrightarrow B$ vector bundle

$$\mathrm{Th}(V) := \Sigma^\infty \left(\frac{V}{V \setminus \{0\}} \right) \in \mathrm{Pic}(SH(B))$$

$$\mathrm{Th}(\mathbb{A}^n_B) \simeq \mathbb{S}_B^{2n,n}$$

$$\mathrm{Th}(-V) := \mathrm{Th}(V)^{\otimes(-1)}.$$

iii) Morel-Voevodsky purity

$$\begin{tikzcd} Z \arrow[rr, hook] \arrow[dr, "p", "sm"'] & & X \arrow[dl, "sm"] \\ & B & \end{tikzcd}$$

$$SH(Z) \xrightarrow{p_\#} SH(B)$$

$$\Sigma^\infty \left(\frac{X}{X \setminus Z} \right) \simeq p_\# \overbrace{\mathrm{Th}(N_{Z/X})}^{\in SH(Z)}$$

part of the
rich functoriality of $SH(-)$: "six operations"

iii) Cohomology theories

A motivic spectrum $E \in SH(B)$ represents a bigraded cohomology theory on $Sm/_B$:

$$E^{p,q}(X) := SH(B)\left(\Sigma^{\infty}_{\mathbb{P}^1} X_+ \, , \, S^{p,q} \wedge E\right)$$

Motivic spectrum	Cohomology theory
$H\mathbb{Z}$ $\left(\begin{matrix} B = \mathrm{Spec}(k) \\ k \text{ perfect} \end{matrix}\right)$	Motivic cohomology $\simeq$ Higher Chow groups
KGL	Homotopy-invariant algebraic K-theory
$H\widetilde{\mathbb{Z}}$ $\left(\begin{matrix} B = \mathrm{Spec}(k) \\ k \text{ perfect} \end{matrix}\right)$	Milnor-Witt motivic cohomology $\simeq$ "Higher Chow-Witt groups"

KO

Homotopy-invariant hermitian K-theory

Orientations and characteristic classes

. $V \longrightarrow B$ vector bundle.

$Th(V)$ truly depends on V.

However, in some cohomology theories the situation simplifies:

- $$\begin{cases} Th(V) \wedge H\mathbb{Z} \simeq \mathbb{S}^{2r,r} \wedge H\mathbb{Z} \\ Th(V) \wedge KGL \simeq \mathbb{S}^{2r,r} \wedge KGL \end{cases}$$

We say that $H\mathbb{Z}$, KGL are (GL-)oriented.

$$\begin{cases} Th(V) \wedge \widetilde{H\mathbb{Z}} \simeq \mathbb{S}^{2r,r} \wedge Th(\det V) \wedge \widetilde{H\mathbb{Z}} \\ Th(V) \wedge KO \simeq \mathbb{S}^{2r,r} \wedge Th(\det V) \wedge KO \end{cases}$$

We say that $\widetilde{H\mathbb{Z}}$, KO are SL-oriented.

- This has concrete consequences:

  - If E is oriented, there is a theory of Chern classes:

    $\forall\ 0 \leq i \leq r, \quad c_i(V) \in E^{2i,i}(B)$

    with properties very similar to the Chern classes in CH^*.

  - If E is SL-oriented, we can twist the associated cohomology theory by a line bundle L.

$$E^{p,q}(X,L) := E^{p+2,q+1}(\mathrm{Th}(L))$$

and there is then a Euler class

$$e(V) \in E^{2r,r}(X, \det(V)^{-1})$$

Link with $GW(k)$

$SH(B)$ combines the notorious simplicity of stable homotopy theory and motives!

How can we extract reasonable invariants?

Thm (Morel) Let k be a perfect field.

$$\mathrm{End}_{SH(k)}(\mathbb{S}^{0,0}) \simeq GW(k).$$

(The map $\longleftarrow$ is very simple: $\langle a \rangle \in GW(k)$ is sent to $[g_a] \in \mathrm{End}_{SH(k)}(\Sigma^\infty(\mathbb{P}^1,\infty)) \simeq \mathrm{End}(\mathbb{S}^{0,0})$

where $g_a : \mathbb{P}^1 \to \mathbb{P}^1,\ [x,y] \mapsto [ax,y]$)

· The unit maps:

$$\begin{cases} \mathbb{S}^{0,0} \longrightarrow H\widetilde{\mathbb{Z}} \\ \mathbb{S}^{0,0} \longrightarrow KO \end{cases}$$

induce

⚠ maps don't induce maps on End; correct this if reused

$$GW(k) = \mathrm{End}(\mathbb{S}^{0,0}) \xrightarrow{\sim} \mathrm{End}(H\widetilde{\mathbb{Z}}) \simeq \widetilde{CH}_0(k) \simeq GW(k)$$

$$\searrow^{\sim}\ \mathrm{End}(KO) = KH_0(k) \simeq GW(k)$$

· The unit maps:

$$\begin{cases} \mathbb{S}^{0,0} \longrightarrow H\mathbb{Z} \\ \mathbb{S}^{0,0} \longrightarrow KGL \end{cases}$$

induce only

$$\mathrm{rnk} : GW(k) \longrightarrow \mathbb{Z}$$

$\rightsquigarrow$ all the quadratic information is lost.

$\rightsquigarrow$ need $H\widetilde{\mathbb{Z}}$, KO for quadratic enumerative geometry.

$\mathrm{End}(\mathbb{S}^{0,0})$ is the receptacle for traces in $SH(k)$.

Traces

C symmetric monoidal category.

$X \in C$ is strongly dualizable if there is $X^\vee \in C$

and $\begin{cases} ev : X \otimes X^\vee \longrightarrow \mathbb{1} \\ coev : \mathbb{1} \longrightarrow X \otimes X^\vee \end{cases}$ satisfying ...

. Let $f \in \mathrm{End}(X)$. We can form

$$\begin{tikzcd} \mathbb{1} \arrow[r, "coev"] \arrow[rrr, bend right, "tr(f)"'] & X \otimes X^\vee \arrow[r, "f \otimes id"] & X \otimes X^\vee \arrow[r, "ev"] & \mathbb{1} \end{tikzcd}$$

$tr(f)$ trace of f

Traces in SH

Thm

i) (Ayoub) $X \xrightarrow{p} B$ smooth projective

Then $\Sigma^\infty X_+$ is strongly dualizable, and

$$(\Sigma^\infty_+ X)^\vee \simeq p_\# \mathrm{Th}(-T_{X/B})$$

ii) (Riou) k perfect field, $\mathrm{char}(k) = p \geq 0$

Then any object in

$$SH_c(k) = \langle \Sigma^\infty_+ X \mid X \in Sm_k \rangle^{df}$$

is strongly dualizable (in $SH(k)[\frac{1}{p}]$)

Def Let k be any field of char $\neq 2$, and $M \in SH_c(k)$. The quadratic Euler characteristic of M is $\chi^{GW}(M) := \mathrm{tr}(\mathrm{id}_M) \in GW(k)$.

$$\left(\overset{\text{IS}}{GW(k_{perf})} \right)$$

- In particular, one can define a (compactly supported) quadratic Euler characteristic for any $X \xrightarrow{f} \mathrm{Spec}(k)$ finite type separated:

$$\chi^{GW}_{(c)}(X) := \chi^{GW}(\underset{(*)}{f_!} f^! \mathbb{1}^{0,0}) \in GW(k)$$

- χ_c^{GW} satisfies a cut-and-paste formula:

$$Z \hookrightarrow X \hookleftarrow U$$

$$\Downarrow$$

$$\chi_c^{GW}(X) = \chi_c^{GW}(Z) + \chi_c^{GW}(U)$$

Thm (Levine-Raksit; "motivic Gauss-Bonnet")

$\begin{cases} k \text{ perfect of char } \neq 2 \\ X \xrightarrow{p} \mathrm{Spec}(k) \text{ be smooth projective.} \end{cases}$

Then $\chi^{GW}(X) = p_* e(T_{X/k})$ in KH

$= \mathrm{Hdg}(X/k) = \chi_{coh}^{GW}(X)$

Hypersurfaces

$F \in k[x_0, \dots, x_{n+1}]_e$, $e > 1$ prime to char(k)

$$X = V(F) \subseteq \mathbb{P}^{n+1}_k \quad \text{smooth hypersurface}$$

$$J(F) := \frac{k[x_0, \dots, x_{n+1}]}{\left(\frac{\partial F}{\partial x_i}\right)_{0 \leq i \leq n+1}} \quad \text{Jacobian ring.}$$

- $J(F)$ is a graded Gorenstein algebra, with socle $J(F)_{(e-2)(n+2)}$ which has a canonical generator e_F, the **Scheja-Storch form.**

$$e_F \in J(F)_{(e-2)(n+2)}$$

(We have $e_F = \frac{1}{\dim J(F)} \cdot \mathrm{Hess}(F)$.

(when this makes sense))

- We get a canonical non-degenerate symmetric bilinear form

$$B_{Jac} : J(F) \times J(F) \longrightarrow k$$

with $B_{Jac}(x,y) = \begin{cases} \lambda, & xy = \lambda e_F \\ 0, & \text{otherwise} \end{cases}$

<u>Thm</u> (Carlson-Griffiths, Dolgachev, LPLS)

i) We have

$$\mathrm{Hdg}(X/k) = \begin{cases} -e\, B_{Jac} \perp \langle e \rangle, & n \text{ even} \\ -e\, B_{Jac}, & n \text{ odd} \end{cases}$$

ii) Analoguous statement for hypersurfaces in a weighted projective space.

<u>Idea</u>: Relies on Griffths's identification of the primitive (Hodge) cohomology of X via residues of forms on $\mathbb{P}^{n+1} \setminus X$.

$$
\begin{tikzcd}
k[x_0, \dots, x_{n+1}]_{(i+1)e-n-2} \arrow[d] & A \arrow[d, mapsto] \\
H^0(\mathbb{P}^{n+1}, \Omega^{n+1}((i+1)X)) \arrow[d, "\mathrm{res}"] & \frac{A}{F^{i+1}} \cdot \sum_{i=0}^{n+1} (-1)^i x_i \, d\hat{x}_i \\
H^0(X, \Omega^n_X(iX)) \arrow[d, "\delta"] & \\
H^i(X, \Omega^{n-i}_X) &
\end{tikzcd}
$$

Deligne - Milnor formula

- Want to understand $X^{(GW)}$ beyond the smooth projective case; looking at a smooth variety degenerating into an hypersurface singularity.

- Set-up: $S = \mathrm{Spec}(R)$, R discrete valuation ring.

  $\eta \in S$ generic point, $\sigma \in S$ closed point

  $t \in R$ fixed uniformizer

  $\mathfrak{X} \xrightarrow{f} S$ flat, finite type, separated of relative dim. n

  $\mathfrak{X}$ regular, $\mathfrak{X}_{\eta}/_{\eta}$ smooth

  (for convenience)

  $\mathfrak{X}_{\sigma}/_{\sigma}$ smooth outside of one point $x_0 \in \mathfrak{X}_{\sigma}(k(\sigma))$

- We have the Milnor number (also for convenience)

$$\mu(\mathfrak{X}, x_0) := \dim_{k(x_0)} \mathcal{E}xt^1(\Omega^1_{\mathfrak{X}/S}, \mathcal{O}_{\mathfrak{X}})_{x_0} < \infty$$

- Fix a system of local parameters z_i on $\mathfrak{X}$ around x_0.

$$J(\mathfrak{X}, x_0) := \mathcal{O}_{\mathfrak{X},x_0} \Big/ \left(\frac{\partial(tf)}{\partial z_i}\right)$$

(local) Jacobian ring

Then $\mu(\mathfrak{X}, x_0) = \dim_{k(x_0)} J(\mathfrak{X}, x_0)$

Thm (Milnor / $\mathbb{C}$, Deligne)

Suppose S is of equal char.

ℓ-adic vanishing cycles

$= 0$ in char 0

i) $(-1)^n \mu(\mathfrak{X}, x_0) = \dim\left(\phi_t(\mathbb{Q}_\ell)_{x_0}\right) + \text{Swan}$

$\quad = \chi(F_{x_0})$

ii) Assume moreover f proper. Then

$$(-1)^n \mu(\mathfrak{X}, x_0) = \chi(X_\eta) - \chi(X_\sigma) + \text{Swan}$$

Rmk: • Still open for S of mixed characteristic.

• i) $\Rightarrow$ ii)

Quadratic refinements

• Let us focus on $\begin{cases} \text{the global formula } (f \text{ proper}). \\ \text{equal char} \end{cases}$

$$\begin{cases} \chi^{GW}(X_\eta) \in GW(k(\eta)) \\ \chi^{GW}(X_\sigma) \in GW(k(\sigma)) \end{cases}$$

different rings!

- Specialisation map

$$sp_t : GW(k(\eta)) \longrightarrow GW(k(\sigma))$$

unique ring hom with

$$\begin{cases} a \in \mathcal{O}_S^\times \Rightarrow sp_t(\langle a \rangle) = \langle \bar{a} \rangle \\ sp_t(\langle t \rangle) = \langle 1 \rangle \end{cases}$$

We can thus form

$$sp_t\, \chi^{GW}(X_\eta) - \chi^{GW}(X_\sigma) \in GW(k(\sigma))$$

- On the other side, using the theory of localized Euler classes, Kass-Wickelgren / Bachmann-Wickelgren define a quadratic Milnor number (or form)

$$\mu^{GW}(\mathfrak{X}, x_0, t) \overset{= e(\Omega^1_{\mathfrak{X}/k},\, df,\, \underline{t})}{\in} GW(k(x_0))$$

$$k(x_0) \overset{\shortparallel}{=} k(\sigma)$$

- In fact $p^{GW}(\mathfrak{X}, x_0, t)$ is given by a Scheja-Storch form on the local Jacobian ring $J(\mathfrak{X}, x_0)$.

Q i) Is there a quadratic refinement of the Swan conductor?

ii) In situations where Swan $= 0$, does the D-M formula lift to GW?

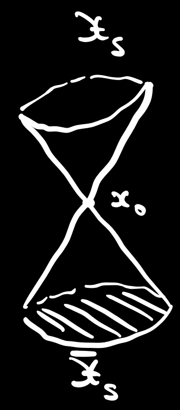
A i) ? ? ?

ii) No !

Thm

i) (LPLS) e prime to $\mathrm{char}(k)$

$\mathfrak{X} \subseteq \mathbb{P}^{n+1}_S$ hypersurface defined by



$$F(x_0, \dots, x_n) - t\, X_{n+1}^e = 0$$

with $\overline{\mathfrak{X}}_s := \{F = 0\} \subseteq \mathbb{P}^n_k$ smooth

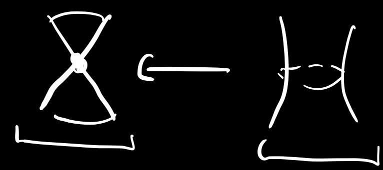
Then $$(\langle e\rangle - \langle 1\rangle) + \underbrace{(\langle e\rangle)^n (-1)^n}_{(-\langle e\rangle)^n} \mu(\mathfrak{X}, x_0, t)$$

$$= sp_t\, \chi^{GW}(X_\eta) - \chi^{GW}(X_\sigma)$$

i') (LPLS) Analogous statement with hypersurfaces in weighted projective space.

ii) (LPLS; Ran Azouri)$^{\mathrm{char}(k) = 0}$ Local version of i) – i'):

eg for i), can assume finitely many isolated singularities, each "resolved by a single blow-up."



Proof:

i) - i'): Direct computation of both sides,

using : - cut and paste for χ_c^{GW}

- formula for χ^{GW} for smooth proj hypersurfaces
- comparison of local and global Jacobian algebras and Scheja-Storch forms.

ii): $\quad \Phi_t : SH(\mathfrak{X}) \to SH(\mathfrak{X}_\sigma)$

LPLS:

$$sp_t\, \chi^{GW}(X_\eta) - \chi^{GW}(X_\sigma) = \chi^{GW}\left(\Phi_t(\mathbb{S}^{0,0})_{x_0}\right)$$

with $\Phi_t(\mathbb{S}^{0,0})_{x_0} \in SH(k(x_0))$ Ayoub's motivic vanishing cycles.

Azouri:

Globalise using

- relation between $\Phi_t(\mathbb{S}^{0,0})$ and Denef-Loeser mot. integration. (Ayoub-Ivorra-Sebag)

- Computation in motivic integration. $\square$

<u>Conclusion</u>: • Mysterious correction terms, no guess yet for the general case.