Motivic infinite loop spaces

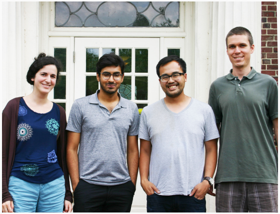
June 11, 2020

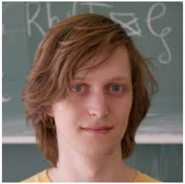
Our topic

- Recent development in motivic homotopy theory: Recognition principle for infinite $\mathbb{P}^1$-loop spaces.
- Result of *M. Yakerson, A. Khan, E. Elmanto, M. Hoyois and V. Sosnilo.* ("Motivic infinite loop spaces" `https://arXiv.org/abs/1711.05248`)
- Uncompromisingly ∞-categorical! Probably difficult to do otherwise in this particular approach, but see `https://arxiv.org/abs/1907.00433` for another point of view.
- Strongly based on results by A. Ananyevskiy, G. Garkusha, A. Neshitov, and I. Panin, developing a fundamental insight of V. Voevodsky.

Dramatis Personae

Yakerson, Khan, Elmanto, Hoyois, Sosnilo; Voevodsky





Ananyevskiy, Garkusha, Neshitov, Panin

Overview

- Motivic homotopy theory
- Infinite loop spaces in topology
- Main results
- Framed correspondences and framed transfers
- Elements of the proof

Motivic homotopy theory

Why motivic homotopy theory?

- Empirical observation: many cohomological invariants of smooth algebraic varieties satisfy
  - Descent for the Nisnevich topology
  - $\mathbb{A}^1$-homotopy invariance
  - Some form of Poincaré duality
- Examples:
  - Algebraic K-theory (Grothendieck, Quillen, Thomason,...)
  - Chow groups and higher Chow groups (Grothendieck, Fulton-MacPherson, Bloch, Levine,...)
  - Hermitian K-theory (Karoubi, Schlichting, Hornbostel,...)
  - Algebraic cobordism (Levine, Morel)
- Morel and Voevodsky introduced a unified framework:
  - Unstable and stable motivic homotopy categories
  - Generalized motivic cohomology theories

Algebraic K-theory

- $\mathbf{Proj}(R)$ category of projective finite rank R-modules over a ring R. The groupoid $\mathbf{Proj}(R)^{\simeq}$ is a commutative monoid under $\oplus$.

$$\text{Grothendieck group:} \quad K_0(R) := (\pi_0(\mathbf{Proj}(R)^{\simeq}))^{\mathrm{grp}}$$

- "Homotopically correct" group completion $\Rightarrow$

$$\text{K-theory space:} \quad K(R) := (\mathbf{Proj}(R)^{\simeq})^{\mathrm{grp}} \in \mathbf{Spc}$$

$$\text{(Quillen) K-theory groups:} \quad K_n(R) := \pi_n(K(R), [0]).$$

- Serre-Swan: $\mathbf{Proj}(R) = \mathbf{Vect}(\mathrm{Spec}(R))$.
- K-theory space $K(X)$ for any scheme X (Quillen, Thomason). For each $V \in \mathbf{Vect}(X)$, get a point $[V] \in K(X)$.
- More generally, if $P \in D_{\mathrm{QCoh}}(X)$ is a perfect complex (Zariski locally quasi-isomorphic to a bounded complex of vector bundles), we have $[P] \in K(X) := K(\mathbf{Perf}(X))$.

Nisnevich topology

- The Nisnevich topology is a Grothendieck topology on schemes.
- Zariski $\subset$ Nisnevich $\subset$ Étale.
- Shares good properties of both: finite homotopical dimension, good local structure results.
- An étale morphism $f : U \to X$ is a Nisnevich cover if for all points $x \in X$, there exists a point $u \in f^{-1}(x)$ with $\kappa(u) \simeq \kappa(x)$.
- Equivalently: there is a filtration $\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X_n = X$ by closed subschemes such that f has a section over each $X_i \setminus X_{i-1}$.
- The points of the Nisnevich topos are henselian local rings (Zariski: local rings, étale: strictly henselian local rings).

Nisnevich sheaves

- S noetherian finite dimensional scheme (e.g. variety over a field).

- A presheaf of spaces $F \in \mathbf{PSh}(\mathrm{Sm}/S)$ is a Nisnevich sheaf iff $F(\emptyset) = *$ and F sends distinguished Nisnevich squares:

$$\begin{array}{ccc} W & \hookrightarrow & U \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p}\ \text{étale} \\ V & \underset{\text{open}}{\hookrightarrow} & X \end{array} \qquad \text{satisfying} \qquad p^{-1}(X \setminus V)_{\mathrm{red}} \simeq (X \setminus V)_{\mathrm{red}}$$

  to pullback squares.

- Representable presheaves are (étale hence) Nisnevich sheaves.

- Étale sheaf $\sim$ "Nisnevich sheaf + Galois descent."

- Thomason: Algebraic K-theory is a Nisnevich sheaf (but not étale!)
  $K_2(\mathbb{C})$ is uniquely divisible, $K_2(\mathbb{R})$ has a 2-torsion element $\langle -1, -1 \rangle$
  $\Rightarrow K_2(\mathbb{R}) \neq K_2(\mathbb{C})^{\mathbb{Z}/2} \Rightarrow K(\mathbb{R}) \neq K(\mathbb{C})^{h\mathbb{Z}/2}$.

$\mathbb{A}^1$-invariance and non-$\mathbb{A}^1$-invariance in algebraic geometry

- Let X be a reduced scheme. Then $\mathcal{O}^\times(X) \simeq \mathcal{O}^\times(X \times \mathbb{A}^1)$. This is false for non-reduced schemes:

$$(k[t,\epsilon]/(\epsilon^2))^\times = k^* \times k[t]\epsilon \neq k^* \times k\epsilon = (k[\epsilon]/(\epsilon^2))^\times.$$

- Let X be a normal scheme. Then $\mathrm{Pic}(X) \simeq \mathrm{Pic}(X \times \mathbb{A}^1)$. This is false for non-normal schemes, e.g. the cusp $y^3 = x^2$.

- Let $X = \mathrm{Spec}(R)$ with R a regular k-algebra. Then

$$\mathrm{Vect}(X) \simeq \mathrm{Vect}(X \times \mathbb{A}^1) \text{ (Quillen-Suslin, Popescu)}$$

  This is false for $X = \mathbb{P}^1$!

- Let X be a regular scheme. Then $K(X) \simeq K(X \times \mathbb{A}^1)$ and $\mathrm{CH}^*(X) \simeq \mathrm{CH}^*(X \times \mathbb{A}^1)$. Many counter-examples for singular schemes.

$\mathbb{A}^1$-invariant presheaves

- A presheaf $F \in \mathbf{PSh}(\mathrm{Sm}/S)$ is called $\mathbb{A}^1$-invariant if for all $X \in \mathrm{Sm}/S$, the projection $X \times \mathbb{A}^1 \to X$ induces

$$F(X) \simeq F(X \times \mathbb{A}^1).$$

- Representable presheaves are not always $\mathbb{A}^1$-invariant.
Ex: $\mathbb{G}_m$, projective curves of genus ≥ 1. C-ex: $\mathbb{A}^1, \mathbb{P}^1$.

- If S is regular, then $\mathrm{Pic}(-)$, $K(-)$, $\mathrm{CH}^*(-)$ are $\mathbb{A}^1$-invariant on Sm/S.

$\mathbb{A}^1$-localisation

- Cosimplicial algebraic simplex:

$$\Delta^\bullet_S := \mathrm{Spec}(\mathcal{O}_S[t_0, \ldots, t_\bullet]/(\sum t_i - 1)) \ \ (\simeq \mathbb{A}^\bullet_S).$$

- Suslin-Voevodsky : the localisation functor $L_{\mathbb{A}^1} : \mathbf{PSh}(\mathrm{Sm}/S) \to \mathbf{PSh}^{\mathbb{A}^1}(\mathrm{Sm}/S)$ is the algebraic singular complex:

$$L_{\mathbb{A}^1}(F) = |\underline{\mathrm{Hom}}(\Delta^\bullet_S, F)| = U \mapsto |F(U \times_S \Delta^\bullet_S)|$$

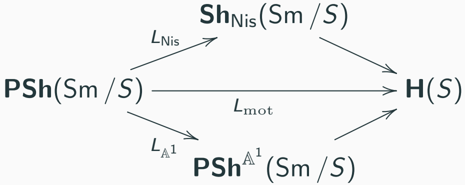
This uses that $\mathbb{A}^1_S$ is an interval object (with multiplication): we have

$$m : \mathbb{A}^1_S \times_S \mathbb{A}^1_S \to \mathbb{A}^1_S, \ \ i_0, i_1 : S \to \mathbb{A}^1_S$$

satisfying some identities.

Unstable motivic homotopy theory

- A motivic space over S is an $\mathbb{A}^1$-invariant Nisnevich sheaf on Sm/S.
- $\mathbf{H}(S) := \mathbf{Sh}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathrm{Sm}/S)$ ∞-category of motivic spaces; we have localisations



- $L_{\mathrm{mot}} = \mathrm{colim}_{n\in\mathbb{N}}(L_{\mathrm{Nis}}L_{\mathbb{A}^1})^{\circ n}$ is very inexplicit...
- L_{mot} is accessible, commutes with finite products, *not* left-exact
  $\mathbf{H}(S)$ presentable, has universal colimits, *not* a ∞-topos.

Motivic spheres

- A motivic equivalence in $\mathbf{PSh}(\mathrm{Sm}/S)$ is a map inverted by L_{mot}.
  Ex: Nisnevich equivalences, naive $\mathbb{A}^1$-homotopy equivalences.

- Pointed motivic homotopy category $\mathbf{H}_\star(S) := \mathbf{H}(S)_{S/-}$

- Bigraded motivic spheres $\mathbb{S}^{a,b} := S^{a-b} \wedge (\mathbb{G}_m, 1)^{\wedge b}$ for $a \geq b \geq 0$.

- By Zariski descent and $\mathbb{A}^1$-invariance:

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array} \quad \text{induces in } \mathbf{H}_\star(S) \quad \begin{array}{ccc} (\mathbb{G}_m, 1) & \longrightarrow & \star \\ \downarrow & \ulcorner & \downarrow \\ \star & \longrightarrow & (\mathbb{P}^1, \infty) \end{array}$$

  We deduce:

$$(\mathbb{P}^1, \infty) \simeq S^1 \wedge (\mathbb{G}_m, 1) = \mathbb{S}^{2,1}$$

  and also

$$(\mathbb{P}^1, \infty) \simeq \frac{\mathbb{A}^1}{\mathbb{A}^1 \setminus 0}$$

Thom spaces and purity theorem

- Thom space $\mathrm{Th}_X(V) := \frac{V}{V \setminus 0} \simeq \frac{\mathbb{P}(V \oplus \mathcal{O})}{\mathbb{P}(V)}$ of a vector bundle $V \to X$.

Theorem (Morel-Voevodsky purity)

$Y \subset X$ closed immersion of smooth S-schemes. Motivic equivalence:

$$\frac{X}{X \setminus Y} \simeq \mathrm{Th}_Y(N_{Y/X}).$$

- Analogue of tubular neighbourhood theorem in topology.
- First serious use of Nisnevich (rather than just Zariski) descent.
- Key geometric fact is "implicit function theorem":
  Zariski locally on X, there is an étale morphism $X \to \mathbb{A}^n_S$ such that $Y \subset X$ is pulled back from a linear inclusion $\mathbb{A}^{n-c}_S \subset \mathbb{A}^n_S$.

Homotopy sheaves

- $\mathbb{A}^1$-connected components $\pi_0^{\mathbb{A}^1}(F) \in \mathbf{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/S, \mathbf{Set})$:

$$\pi_0^{\mathbb{A}^1}(F) := L_{\mathrm{Nis}}(U \mapsto \pi_0(L_{\mathrm{mot}}(F)(U)))$$

  and $\mathbb{A}^1$-homotopy sheaves $\pi_n^{\mathbb{A}^1}(F, x_0) \in \mathbf{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/S, \mathbf{Grp}/\mathbf{Ab})$

$$\pi_n^{\mathbb{A}^1}(F, x_0) := L_{\mathrm{Nis}}(U \mapsto \pi_n(L_{\mathrm{mot}}(F)(U), x_0))$$

- $\mathbb{A}^1$-homotopy sheaves detect motivic equivalences.
- Morel: for $S = \mathrm{Spec}(k)$ infinite perfect field, $\pi_n^{\mathbb{A}^1}$ for $n \geq 1$ have very nice properties, “controlled by their values on fields”.
- Slogan: $H(k)$ behaves like an ∞-topos “modulo” $\pi_0^{\mathbb{A}^1}$.

Stable motivic homotopy theory

- Stable motivic homotopy category:

$$\mathbf{SH}(S) := \mathbf{H}_\star(S)[(\mathbb{P}^1,\infty)^{\wedge,(-1)}] \simeq \mathbf{Spt}_{(\mathbb{P}^1,\infty)}\mathbf{H}_\star(S)$$

  $\mathbf{SH}(S)$ is a stable presentable symmetric monoidal ∞-category.

  $E = (E_i)_{i\in\mathbb{N}} \in \mathbf{H}_\star(S)^{\mathbb{N}}$, $\tau_i : E_i \simeq \Omega^1_{\mathbb{P}^1} E_{i+1} := \underline{Hom}_{\mathbf{H}_\star(S)}((\mathbb{P}^1,\infty), E_{i+1})$.

- We have an adjunction

$$\Sigma^\infty_{\mathbb{P}^1} : \mathbf{H}_\star(S) \leftrightarrows \mathbf{SH}(S) : \Omega^\infty_{\mathbb{P}^1}$$

- Bigraded generalised motivic cohomology theories for $E \in \mathbf{SH}(S)$:

$$E^{a,b} : \mathrm{Sm}/S \to \mathbf{Ab}, \quad X \mapsto [\Sigma^\infty_{\mathbb{P}^1} X_+, \mathbb{S}^{a,b} \otimes E].$$

Realisation functors

- Let $\sigma : k \to \mathbb{C}$ be a field embedding. Complex Betti realisation:

$$R_{B,\sigma} : \mathbf{SH}(k) \to \mathcal{SH}, \quad \Sigma^\infty_{\mathbb{P}^1} X_+ \mapsto \Sigma^\infty_{S^1} |X_\sigma(\mathbb{C})|_+$$

- Let $\iota : k \to \mathbb{R}$ be a field embedding. Real Betti realisation:

$$R_{B,\iota} : \mathbf{SH}(k) \to \mathcal{SH}(C_2), \quad \Sigma^\infty_{\mathbb{P}^1} X_+ \mapsto (\Sigma^\infty_{S^1} |X_\iota(\mathbb{C})|_+, \mathrm{conj})$$

- Slogan (Morel): motivic homotopy theory in char. 0 “mixes” the homotopy theory of complex and real points.

Some motivic ring spectra

$$\mathbb{S}_k \to \mathbf{MGL} \to \mathbf{KGL}, \mathbf{M}\mathbb{Z}$$

motivic analogue of the sequence

$$\mathbb{S} \to \mathbf{MU} \to \mathbf{KU}, \mathbf{H}\mathbb{Z}.$$

(For $k = \mathbb{C}$, the top line realises to the bottom line)

- Motivic sphere spectrum $\mathbb{S}_k$
- Algebraic cobordism spectrum
  $\mathbf{MGL} = (\mathrm{Th}(V_n))_{n\in\mathbb{N}}$ with $V_n \to \mathrm{Gr}(n, \infty)$.
- Algebraic K-theory spectrum
  $\mathbf{KGL} = (\mathbb{Z} \times \mathrm{Gr}(\infty,\infty), \mathbb{Z} \times \mathrm{Gr}(\infty,\infty), \ldots)$
- Motivic cohomology spectrum (à la Dold-Thom)
  $\mathbf{M}\mathbb{Z} = (\mathrm{Spec}(k)_+, \mathrm{Sym}^\infty \mathbb{P}^1, \ldots, \mathrm{Sym}^\infty(\mathbb{P}^1)^{\wedge n}, \ldots)$ $(\mathrm{char}(k) = 0)$
- More: Hermitian K-theory, Morava K-theories, **MSL**, **MSp**, etc.

Some representability results

- Let $X \in \mathsf{Sm}/k$ with k perfect. Then

$$\begin{aligned}\mathbf{M}\mathbb{Z}^{a,b}(X) &= H^a_M(X, \mathbb{Z}(b)) \quad \text{(motivic cohomology groups, via } \mathbf{DM}\text{)} \\ &= \mathrm{CH}^b(X, 2b-a) \quad \text{(higher Chow groups)}\end{aligned}$$

- Let X be a regular scheme. Then

$$\mathbf{KGL}^{a,b}(X) = K_{2b-a}(X)$$

- Canonical isomorphism

$$\mathbf{KGL} \otimes \mathbf{M}\mathbb{Q} \simeq \bigoplus_{n \in \mathbb{Z}} \Sigma^{2n,n}\mathbf{M}\mathbb{Q}$$

lifting the Chern character from K-theory to higher Chow groups; can also lift the Grothendieck-Riemann-Roch theorem to **SH** (Riou).

Functoriality of SH

- Voevodsky, Ayoub: $\mathbf{SH}(-)$ admits a six operation formalism ("parametrized motivic homotopy theory").
- $f : X \to S$ finite type morphism of schemes.

$$f^* : \mathbf{SH}(S) \leftrightarrows \mathbf{SH}(X) : f_*$$

$$f_! : \mathbf{SH}(X) \leftrightarrows \mathbf{SH}(S) : f^!$$

  satisfying base change, projection formula, Atiyah-Verdier duality... Close analogy with theory of constructible/ℓ-adic sheaves.
- Exceptional pushforward $f_!$ is determined by $p_! = p_*$ for p proper by the cofiber localisation sequence for an open/closed pair (j, i):

$$j_! j^! \to \mathrm{id} \to i_* i^*$$

- For $f : X \to S$ smooth, we have $\Sigma^\infty_{\mathbb{P}^1} X_+ = f_! f^! \mathbb{S}$ in $\mathbf{SH}(S)$.

Thom spectra and motivic J-homomorphism

- Let $p : V \to X$ vector bundle. The Thom spectrum

$$\mathbf{Th}_X(V) := \Sigma^\infty_{\mathbb{P}^1}\left(\frac{V}{V \setminus 0}\right) \in \mathbf{SH}(X).$$

  is $\otimes$-invertible in $\mathbf{SH}(X)$ (recall that $\mathbb{A}^1/(\mathbb{A}^1 \setminus 0) \simeq (\mathbb{P}^1, \infty)$).

- Extends to perfect complexes and factors through K-theory:

$$\mathbf{Th}_X : K(X) \to \mathrm{Pic}(\mathbf{SH}(X)),\ [V] \mapsto \mathbf{Th}_X(V)$$

  Gives further twists for motivic cohomology theories: for $\xi \in K(X)$,

$$E^{a,b}(X, \xi) := [\mathbf{Th}_X(\xi), \mathbb{S}^{a,b} \otimes E]$$

Transfers for smooth and finite étale morphisms

- Motivic Atiyah duality (Ayoub): let $f : X \to Y$ be a smooth projective morphism in Sm/S. We have in $\mathbf{SH}(Y)$:

$$\begin{aligned}(\Sigma^\infty_{\mathbb{P}^1} X_+)^\vee &= (f_! f^! \mathbb{S})^\vee \\ &= f_* f^* \mathbb{S} \text{ (duality exchanges operations)} \\ &= f_! f^* \mathbb{S} \text{ (f proper)} \\ &= f_!(f^! \mathbb{S} \otimes \mathbf{Th}_X(\Omega^1_{X/Y})^{-1}) \text{ (purity for smooth morphisms)}\end{aligned}$$

- For f finite étale ($\Rightarrow \Omega^1_{X/Y} = 0$) we get $(\Sigma^\infty_{\mathbb{P}^1} X_+)^\vee \simeq \Sigma^\infty_{\mathbb{P}^1} X_+$ in $\mathbf{SH}(Y)$. This yields a wrong-way transfer map

$$\Sigma^\infty_{\mathbb{P}^1} Y_+ \simeq (\Sigma^\infty_{\mathbb{P}^1} Y_+)^\vee \xrightarrow{f^\vee} (\Sigma^\infty_{\mathbb{P}^1} X_+)^\vee \simeq \Sigma^\infty_{\mathbb{P}^1} X_+$$

and hence finite étale transfers $E^{a,b}(X) \to E^{a,b}(Y)$ for all generalized motivic cohomology theories.

Infinite loop spaces in algebraic topology

E_∞-spaces à la Segal

- Segal's category $\Gamma := (\mathrm{Fin}_*)^{\mathrm{op}}$. Write $\langle n \rangle := \{1, 2, \dots n\}_+ \in \Gamma$.
- Let C be an ∞-category with finite products. Segal condition on $F \in \mathbf{PSh}(\Gamma, C)$:
$$F\langle n \rangle \xrightarrow{\sim} F\langle 1 \rangle^{\times n}.$$
- Commutative monoids (= cartesian E_∞-algebras):
$$\mathbf{CMon}(C) := \mathbf{PSh}_{\mathrm{Segal}}(\Gamma, C).$$
- In particular get monoid structure on $F\langle 1 \rangle$ in $\mathrm{Ho}(C)$:
$$F\langle 1 \rangle \times F\langle 1 \rangle \xleftarrow{\sim} F\langle 2 \rangle \xrightarrow{1,2 \mapsto 1} F\langle 1 \rangle$$
- $\mathbf{CMon}(\mathbf{Spc})$ is the ∞-category of E_∞-spaces.

Group completion

- $F \in \mathbf{CMon}(C)$ is called grouplike if the monoid $F\langle 1 \rangle$ is a group.
- If finite products distribute over colimits, the inclusion $\mathbf{CMon}(C)^{\mathrm{grp}} \hookrightarrow \mathbf{CMon}(C)$ has a left adjoint, the group completion functor:
$$(-)^{\mathrm{grp}} : \mathbf{CMon}(C) \to \mathbf{CMon}(C)^{\mathrm{grp}}$$
- Ex: $K(X)$ grouplike E_∞-space.

Recognition principle

- $\mathbf{Spt}_{\geq 0} \subset \mathbf{Spt}$ connective spectra : $E = (E_i)_{i \in \mathbb{N}}$ with E_i is i-connective for all $i \Leftrightarrow$ generated under colimits by $\Sigma^\infty \mathbf{Spc}_*$.
- Let $E \in \mathbf{Spt}$. Then the infinite loop space $X_0 = \Omega^\infty E$ has a natural structure of grouplike E_∞-space:

$$X_0\langle n\rangle := \mathrm{Map}_{\mathbf{Spt}}(\mathbb{S}^{\times n}, E).$$

Theorem (Segal 74; Recognition principle)

$$\mathbf{Spt}_{\geq 0} \simeq \mathbf{CMon}(\mathbf{Spc})^{\mathrm{grp}}.$$

Ex: The group-like E_∞-space $K(X)$ yields a K-theory spectrum $\mathbf{K}(X)$.

The recognition principle can be decomposed into:

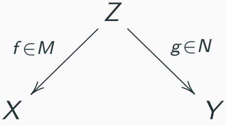
- (Reconstruction) $\mathbf{Spt}(\mathrm{for}) : \mathbf{Spt}(\mathbf{CMon}(\mathbf{Spc})^{\mathrm{grp}}) \simeq \mathbf{Spt}$.
- (Cancellation) The functor

$$\Sigma_{S^1} : \mathbf{CMon}(\mathbf{Spc})^{\mathrm{grp}} \to \mathbf{Spt}(\mathbf{CMon}(\mathbf{Spc}))$$

is fully faithful.

Correspondences

- Let C be a 1-category (resp. ∞)-category with pullbacks, $M, N \subset \mathrm{Arr}(C)$ classes of morphisms satisfying some conditions. We have a $(2,1)$-category (resp. $(\infty,1)$-category) $\mathbf{Corr}(C, M, N)$ of correspondences (or spans):



  Composition is given by pullbacks.
- Functors out of $\mathbf{Corr}(C, M, N)$ encode covariant functoriality $f_!$ in N, contravariant functoriality g^* in M and a (coherent) base change formula:

$$f_! g^* = g'^* f'_!$$

  for cartesian squares $g \circ f = f' \circ g'$.
- Notation $\mathbf{Corr}(C) := \mathbf{Corr}(C, \mathrm{all}, \mathrm{all})$.

E_∞-spaces via correspondences

We have an equivalence $\Gamma \simeq \mathbf{Corr}(\mathbf{Fin}, \mathrm{all}, \mathrm{inj})$:

$$X_+ \mapsto X, \quad (f : X_+ \to Y_+) \mapsto (Y \xleftarrow{f} f^{-1}(Y) \hookrightarrow X).$$

Proposition (Cranch)

Right Kan extension along $\mathbf{Corr}(\mathbf{Fin}, \mathrm{all}, \mathrm{inj}) \subset \mathbf{Corr}(\mathbf{Fin})$ *gives an equivalence*

$$\mathbf{CMon}(C) \simeq \mathbf{PSh}_\Sigma(\mathbf{Corr}(\mathbf{Fin}), C).$$

($\mathbf{PSh}_\Sigma$ = preserves finite products)

Slogan: $* \in \mathbf{Corr}(\mathbf{Fin})$ is the *universal commutative monoid*.

Corollary

$\mathbf{PSh}_\Sigma(\mathbf{Corr}(\mathbf{Fin}), \mathbf{Spc})$ *also models* E_∞*-spaces.*

Recognition principle via correspondences of manifolds

- "Morel-Voevodsky" approach to the category of spaces (Dugger):

$$\mathbf{Spc} \simeq \mathbf{Sh}^{\mathbb{R}}(\mathbf{Mfd}), \quad X \mapsto h_X := \mathrm{Map}(\Pi_\infty(-), X)$$

- For $f : M \to N$ finite covering map in **Mfd**, Atiyah duality gives a transfer map $\Sigma^\infty N_+ \to \Sigma^\infty M_+$.
- For $E \in \mathbf{Spt}$ and $X = \Omega^\infty E$, this yields a map

$$\mathrm{Map}(M, X) = \mathrm{Map}_{\mathbf{Spt}}(\Sigma^\infty M_+, E) \to \mathrm{Map}_{\mathbf{Spt}}(\Sigma^\infty N_+, E) = \mathrm{Map}(N, X).$$

Theorem (Quillen's transfer conjecture; Bachmann-Hoyois)

$$\mathbf{Spt}_{\geq 0} \simeq \mathbf{Sh}^{\mathbb{R}}(\mathbf{Corr}(\mathbf{Mfd}, \mathrm{fcov}, \mathrm{all})), \; E \mapsto h_{\Omega^\infty E}$$

Motivic $\mathbb{P}^1$-infinite loop spaces and framed transfers

Main result: motivic Quillen transfer conjecture

Let k be a perfect field. $\mathbf{SH}^{\mathrm{veff}}(k) \subset \mathbf{SH}(k)$ very effective motivic spectra: generated by $\Sigma^{\infty}_{\mathbb{P}^1} X_+$ under extensions and colimits.

Theorem (Motivic recognition principle; EHKSY, 2017)

Let k be a perfect field. There is a canonical equivalence

$$\mathbf{SH}^{\mathrm{veff}}(k) \simeq \mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{grp}} := \mathbf{Sh}^{\mathbb{A}^1}_{\mathsf{Nis}}(\mathbf{Corr}^{\mathrm{fr}}(k))^{\mathrm{grp}}$$

of symmetric monoidal ∞-categories.

- $\mathbf{Corr}^{\mathrm{fr}}(k)$ is the category of framed correspondences: suitable replacement, to be defined, for $\mathbf{Corr}(\mathbf{Mfd}, \mathrm{fcov}, \mathrm{all})$ in algebraic geometry.
- The same result with $\mathbf{Corr}^{\mathrm{fr}}(k)$ replaced by $\mathbf{Corr}(\mathsf{Sm}/k, \mathrm{f\acute{e}t}, \mathrm{all})$ is not known; need more transfers.

Two auxiliary results

The recognition principle follows from

Theorem (EHKSY; Reconstruction theorem)

Let S be any base scheme. The graph functor $\mathrm{Sm}/S \to \mathbf{Corr}^{\mathrm{fr}}(S)$ *induces an equivalence*

$$\mathbf{SH}(S) \simeq \mathbf{SH}^{\mathrm{fr}}(S) := \mathbf{Spt}_{\mathbb{P}^1}\mathbf{H}^{\mathrm{fr}}(S).$$

Theorem (EHKSY; Cancellation theorem)

Let k be a perfect field. The functor

$$\Sigma^{\infty}_{\mathbb{P}^1} : \mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{grp}} \to \mathbf{SH}^{\mathrm{fr}}(k)$$

is fully faithful.

Cotangent complex

- Want more transfers on motivic infinite loop spaces $\Rightarrow$ looking for morphisms which "behave" like finite étale morphisms.
- A morphism $f : X \to S$ has a cotangent complex $\mathbb{L}_f \in D_{\mathrm{QCoh}}(X)$ which controls its deformation theory.
- There is a canonical map $\mathbb{L}_f \to \Omega^1_{X/S}[0]$; can think of $\mathbb{L}_-$ as the derived functor $\mathbb{L}\Omega^1_{-/S}[0]$.
- Let f be locally of finite presentation. Then f is smooth iff $\mathbb{L}_f = \Omega^1_{X/S}[0]$ and f is étale iff $\mathbb{L}_f = 0$.
- Fundamental cofiber sequence of cotangent complexes:

$$X \xrightarrow{f} Z \xrightarrow{g} Y \;\Rightarrow\; \mathbb{L}_f \to \mathbb{L}_{g \circ f} \to f^* \mathbb{L}_g$$

Local complete intersection morphisms

- In algebraic geometry we often need more equations than codimension to define singular subvarieties/morphisms $\Rightarrow$ notion of local complete intersection.
- A morphism of schemes f is a local complete intersection (lci) morphism if f factors Zariski locally on X, as

$$f : X \xhookrightarrow{i} Z \xrightarrow{p} S$$

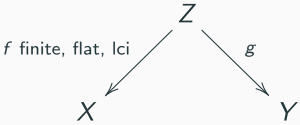
  with p smooth and i closed immersion cut by a regular sequence.
- f lci $\Rightarrow \mathbb{L}_f$ perfect complex with amplitude $[0,1]$: locally, after choosing a factorisation $f = p \circ i$:

$$\mathbb{L}_f = [\mathcal{N}_{X/Z} \to i^*\Omega_{Z/S}].$$

- In particular get K-theory class $[\mathbb{L}_f] \in K(X)$ and associated Thom spectrum.

Framed correspondences

- Let $X, Y \in \mathrm{Sm}/S$. A framed correspondence from X to Y is a correspondence



  together with a path $\alpha : 0 \sim [\mathbb{L}_f]$ in the K-theory space $K(Z)$.
- EHKSY: General construction of ∞-category of labelled correspondences $\mathbf{Corr}^F(C, M, N)$ for F "labelling functor".
- EHKSY construct a labelling functor $\mathbf{K} - \mathbf{triv}(-)$ encoding α using the fundamental cofiber sequence of cotangent complexes and the additivity theorem in K-theory.

$$\mathbf{Corr}^{\mathrm{fr}}(S) := \mathbf{Corr}^{\mathbf{K-triv}}(\mathrm{Sch}_S, \mathrm{fin} + \mathrm{flat} + \mathrm{lci}, \mathrm{all})_{|\,\mathrm{Sm}/S}$$

Framed transfers

- Let $(f, g, \alpha) \in \mathbf{Corr}^{\mathrm{fr}}(X, Y)$. By work of Déglise-Jin-Khan, motivic cohomology theories have twisted transfers for finite lci morphisms $\Rightarrow$ framed transfers:

$$E^{a,b}(X) \xrightarrow{g^*} E^{a,b}(Z) \overset{\alpha}{\simeq} E^{a,b}(Z, \mathbb{L}_f) \xrightarrow{f_! \ (DJK)} E^{a,b}(Y).$$

- This suggests that motivic infinite $\mathbb{P}^1$-loop spaces are group-like motivic spaces with framed transfers:

$$\Omega^{\infty}_{\mathbb{P}^1} : \mathbf{SH}(k) \to \mathbf{Sh}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathbf{Corr}^{\mathrm{fr}}(k))^{\mathrm{grp}}$$

- The Main theorem claims that this is the case and there is an equivalence:

$$\Omega^{\infty}_{\mathbb{P}^1} : \mathbf{SH}^{\mathrm{veff}}(k) \simeq \mathbf{Sh}^{\mathbb{A}^1}_{\mathrm{Nis}}(\mathbf{Corr}^{\mathrm{fr}}(k))^{\mathrm{grp}}.$$

Elements of the proof

Origin story: finite correspondences and mixed motives

- Inspiration comes from Voevodsky's theory of finite correspondences over a perfect field k.
- A finite correspondence $Z \in \mathbf{Corr}_k(X, Y)$ between $X, Y \in \mathrm{Sm}/k$ is a finite linear combination of irreducible subvarieties $Z \subset X \times_k Y$ finite and surjective onto an irreducible component.
- Voevodsky's effective mixed motives: $\mathbf{DM}^{\mathrm{eff}}(k) := \mathbf{Sh}^{\mathbb{A}^1}_{\mathsf{Nis}}(\mathbf{Corr}(k))$
- $\mathbb{P}^1$-stabilisation: $\mathbf{DM}(k) = \mathbf{DM}^{\mathrm{eff}}(k)[(\mathbb{P}^1, \infty)^{\otimes -1}] \xrightarrow{\mathrm{EM}} \mathbf{SH}(k)$.
- Eilenberg-Maclane functor

$$\mathrm{EM} : \mathbf{DM}(k) \to \mathbf{SH}(k)$$

  with $\mathrm{EM}(\mathbb{Z}(0)) = \mathbf{M}\mathbb{Z}$.
- Slogan: $\mathbf{DM}(k)$ is to $\mathbf{SH}(k)$ what $D(\mathbf{Ab})$ is to $\mathcal{SH}$.

Two theorems of Voevodsky on DM

Let k be a perfect field.

Theorem (Strict homotopy invariance, Voevodsky)

$$L_{\mathrm{mot}} = L_{\mathsf{Nis}} L_{\mathbb{A}^1} : \mathbf{PSh}(\mathbf{Corr}(k)) \to \mathbf{DM}^{\mathrm{eff}}(k).$$

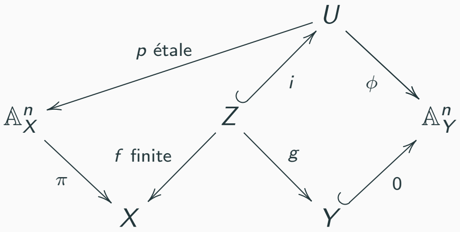
In other words, if $F \in \mathbf{PSh}(\mathbf{Corr}(k))$ is $\mathbb{A}^1$-invariant, then $L_{\mathsf{Nis}} F \in \mathbf{PSh}(\mathbf{Corr}(k))$ is $\mathbb{A}^1$-invariant and hence in $\mathbf{DM}^{\mathrm{eff}}(k)$.

Theorem (Cancellation, Voevodsky)

$$\Sigma^{\infty}_{\mathbb{P}^1} : \mathbf{DM}^{\mathrm{eff}}(k) \hookrightarrow \mathbf{DM}(k) \textit{ is fully faithful.}$$

Equationally framed correspondences

- Framed correspondences are not "geometric" enough to adapt directly the **DM** theory. Need to add some coordinates!
- Let $X, Y \in \mathrm{Sm}/k$. An equationally framed correspondence of level n from X to Y is a diagram

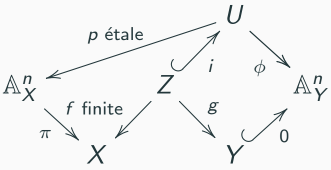


such that U is an étale neighbourhood of Z in $\mathbb{A}^n_X$ and the square is cartesian. The "equation" is $\phi = 0$ cutting out Z.

- Pass to colimit $n \to \infty$ to get: $\mathbf{Corr}^{\mathrm{efr}}(-, Y) : (\mathrm{Sm}/k)^{\mathrm{op}} \to \mathrm{Set}$.

Equationally framed vs framed

In the geometric situation:



- i and $p \circ i$ are regular closed immersions.
- $f = \pi \circ (p \circ i)$ is flat and lci.
- We have $N_{p \circ i} \simeq N_i$ and ϕ induces a trivialisation $N_i \simeq \mathcal{O}_Z^n$.

Since $\mathbb{L}_f = [N_{p \circ i} \to \mathcal{O}_Z^n]$ we get a path $\alpha : [\mathbb{L}_f] \sim 0$ in $K(Z)$. This induces a map of presheaves on $\mathrm{Sm}\,/k$:

$$\mathbf{Corr}^{\mathrm{efr}}(-, Y) \to \mathbf{Corr}^{\mathrm{fr}}(-, Y), \; (Z, U, \phi, g) \mapsto (Z, \alpha).$$

Theorem (EHKSY; "contractibility of spaces of embeddings")

This map is a motivic equivalence.

Framed analogues of Voevodsky's theorems

Theorem (EHKSY+AGNP; strict homotopy invariance)

Let k be an infinite perfect field. Let $F \in \mathbf{PSh}_{\Sigma}(\mathbf{Corr}^{\mathrm{fr}}(k))^{\mathrm{grp}}$.

If F is $\mathbb{A}^1$-invariant, then $L_{\mathrm{Nis}} F$ is $\mathbb{A}^1$-invariant and hence in $\mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{grp}}$.

Theorem (EHKSY+AGNP; Cancellation theorem)

Let k be a perfect field. The functor

$$\Sigma^{\infty}_{\mathbb{P}^1} : \mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{grp}} \to \mathbf{SH}^{\mathrm{fr}}(k)$$

is fully faithful.

With extra work, finishes the proof of Reconstruction and Main theorem.

Framed analogues of Voevodsky's theorems

Theorem (EHKSY+AGNP; strict homotopy invariance)

Let k be an infinite perfect field. Let $F \in \mathbf{PSh}_\Sigma(\mathbf{Corr}^{\mathrm{fr}}(k))^{\mathrm{grp}}$.

If F is $\mathbb{A}^1$-invariant, then $L_{\mathrm{Nis}}F$ is $\mathbb{A}^1$-invariant and hence in $\mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{grp}}$.

Theorem (EHKSY+AGNP; Cancellation theorem)

Let k be a perfect field. The functor

$$\Sigma^\infty_{\mathbb{P}^1} : \mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{grp}} \to \mathbf{SH}^{\mathrm{fr}}(k)$$

is fully faithful.

With extra work, finishes the proof of Reconstruction and Main theorem.

AGNP= Ananyevskiy, Garkusha, Neshitov, and Panin. Work on $\mathbf{Corr}^{\mathrm{efr}}$.

Motivic Barratt-Priddy-Quillen theorem

Barratt-Priddy-Quillen theorem:

$$\Omega^\infty \mathbb{S} \simeq (\mathbf{Fin}^{\simeq})^{\mathrm{grp}}$$

Theorem (EHKSY)

$$\begin{aligned} \Omega^\infty_{\mathbb{P}^1} \mathbb{S}_k &= L_{\mathrm{Nis}} L_{\mathbb{A}^1} \mathbf{Corr}^{\mathrm{fr}}(-, \mathrm{Spec}(k))^{\mathrm{grp}} \\ &= L_{\mathrm{Nis}} (L_{\mathbb{A}^1} \mathbf{Hilb}^{\mathrm{fr}}(\mathbb{A}^\infty))^{\mathrm{grp}} \end{aligned}$$

Similar models for other "motivic Thom spectra."

Framed finite sets:

$$\mathbf{Corr}^{\mathrm{fr}}(X, \mathrm{Spec}(k)) = \{f : Y \to X \text{ finite flat lci} + \alpha : [0] \sim [L_f] \in K(Y)\}$$

Framed Hilbert scheme:

$$\mathbf{Hilb}^{\mathrm{fr}}(X)(T) = \{Z \in \mathbf{Hilb}(X)(T) + \phi : N_{Z/X} \simeq (\Omega^1_{X \times T/T})_{|Z}$$