Motivic mirror symmetry for Higgs bundles

Simon Pepin Lehalleur (RU Nijmegen)

jt work with

Victoria Hoskins (RU Nijmegen)

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Main theorem

C smooth projective curve of genus $g \geq 2$ over $k = \bar{k}$, $\mathrm{char}(k) = 0$.

$(n, d) \in \mathbb{N} \times \mathbb{Z}$ with $(n,d) = 1$, $L \in \mathrm{Pic}^d(C)$, $\Lambda := \mathbb{Q}(\zeta_n)$.

$$\begin{cases} \mathcal{M}_{n,L} = \text{moduli space of semistable } SL_n\text{-Higgs bundles (with det } L) \\ \bar{\mathcal{M}}_{n,d} = \text{moduli orbifold of semistable } PGL_n\text{-Higgs bundles (of degree } d) \end{cases}$$

Thm (Hoskins-PL) In the category $DM(k, \Lambda)$ of Voevodsky motives:

$$M(\mathcal{M}_{n,L}) \simeq M^{orb}(\bar{\mathcal{M}}_{n,d}, \zeta_L)$$

Plan :

I) Moduli of vector and Higgs bundles on curves

II) Topological mirror symmetry after Maulik-Shen

III) Mixed motives & motivic sheaves

IV) Proof of main theorem

I) Moduli of bundles on curves

- Two basic objects : $\begin{cases} \text{moduli } \underline{\underline{\text{stack}}} \ \mathrm{Bun} \text{ of all vector bundles} \\ \text{moduli } \underline{\underline{\text{space}}} \ \mathcal{N} \text{ of semistable vector bundles} \end{cases}$

- $\mathrm{Bun}_{n,d}(C)(T) := \left(V \xrightarrow[\text{rk } n, \deg d]{\text{v.b.}} C \times T \text{ , bundle isomorphisms} \right)$ ← groupoid

  smooth Artin stack of dimension $n^2(g-1)$

- $\mathcal{N}_{n,d}$ "coarse" moduli space of semistable vb of rank n deg d :

  projective variety of dimension $n^2(g-1)+1$, smooth if $(n,d)=1$.

Def: V is semistable if $\forall\, W \underset{\neq 0}{\subsetneq} V$, $\dfrac{\deg W}{\mathrm{rk}\, W} \leq \dfrac{\deg V}{\mathrm{rk}\, V}$

(GL_n-) Higgs bundles (Hitchin)

- Pair (E, θ) with

$$\begin{cases} E & \text{vector bundle of rank } n, \text{ degree } d \text{ on } C \\ \theta : E \xrightarrow{\mathcal{O}_C} E \otimes \omega_C & \text{Higgs field} \end{cases}$$

- Appears via cotangent of Bun :

$$T^*_E \mathrm{Bun}_{n,d} \underset{\text{neglect automorphisms}}{\overset{\text{defn. theory}}{\simeq}} \mathrm{Ext}^1(E,E)^* \overset{\text{Serre duality}}{\simeq} \mathrm{Hom}(E, E \otimes \omega_C)$$

Moduli space of Higgs bundles (Hitchin, Simpson, Nitsure, ...)

$(n,d) = 1$

* $\widetilde{\mathcal{M}}_{n,d}$ moduli space of semi stable Higgs bundles :
  - smooth, quasiprojective, of dimension $2(n^2(g-1)+1)$
  - algebraic symplectic variety $\supset T^*\mathcal{N}_{n,d}$
* Hitchin fibration : $h : \widetilde{\mathcal{M}}_{n,d} \longrightarrow \mathcal{A}_{GL_n} := \bigoplus_{i=1}^{n} H^0(C, \omega_C^{\otimes i})$
  - proper, Lagrangian, with generic fiber over $a \in \mathcal{A}_{GL_n}$

  the Jacobian of the spectral curve

$$\mathcal{C}_a := P_a^{-1}(0) \subseteq \underbrace{T^*C \longrightarrow C}_{n:1} \qquad \left(P_a = \sum a_i t^{n-i}\right)$$

Moduli of Higgs bundles : applications

- Hyperkähler counterpart to $N_{n,d}$ (Hitchin, ...)
- Non-abelian Hodge theory : relation with Betti/de Rham local systems, $P = W$ conjecture (Simpson, Corlette, Hausel, de Cataldo, Migliorini ...)
- Classical limit of the geometric Langlands program (Hausel-Thaddeus, Donagi-Pantev, Kapustin-Witten, ...)
- Affine Springer theory and function field Langlands (Ngo, Yun, ...)

relevant for this talk

SL_n- and PGL_n-Higgs bundles

- Fix $L \in \mathrm{Pic}^d(C)$. A (twisted) SL_n-Higgs bundle is a Higgs bundle with $\begin{cases} \det(E) \simeq L & \text{in } \mathrm{Pic}^d(C) \\ \mathrm{Tr}(\theta) = 0 & \text{in } H^0(C, \omega_C) \end{cases}$

- A PGL_n-Higgs bundle of degree d is ... a special case of the general definition of G-Higgs bundles!

$\left(\mathcal{E}/_C \text{ principal } G\text{-bundle} \ + \ \theta \in H^0(C, \mathrm{Ad}(\mathcal{E}) \otimes \omega_C) \right)$

Moduli of SL_n- and PGL_n-Higgs bundles

$$\begin{cases} \mathcal{M}_{n,L} \hookrightarrow \check{\mathcal{M}}_{n,d} & \text{smooth of codimension } 2g = \begin{matrix} \dim \mathrm{Pic}^d \\ + \\ \dim H^0(C,\omega_C) \end{matrix} \\ h : \mathcal{M}_{n,L} \longrightarrow \mathcal{A}_n := \bigoplus_{i=2}^{n} H^0(C, \omega_C^{\otimes i}) & \text{proper, Lagrangian} \end{cases}$$

$*$ $\mathrm{Jac}(C)$ acts on $\check{\mathcal{M}}_{n,d}$ by tensoring

$\cup$

$\Gamma := \mathrm{Jac}(C)[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$ " " $\mathcal{M}_{n,L}$ " " "

$*$ $\bar{\mathcal{M}}_{n,d} := [\mathcal{M}_{n,L}/\Gamma] \xrightarrow{\bar{h}} \mathcal{A}_n$ same Hitchin base!

II) Topological mirror symmetry

$G, G^\vee$ Langlands dual reductive groups $\Longrightarrow$

$$\begin{tikzcd} \mathcal{M}_G \arrow[dr, "h_G"'] & & \mathcal{M}_{G^\vee} \arrow[dl, "h_G"] \\ & \mathcal{A}_G \simeq \mathcal{A}_{G^\vee} & \end{tikzcd}$$

Thm (Hausel-Thaddeus for $G = SL_n$; Donagi-Pantev in general)

- For generic $a \in \mathcal{A}_G$ the Hitchin fibers $h_G^{-1}(a)$, $h_{G^\vee}^{-1}(a)$ are dual abelian varieties.
- For $G = SL_n$: $\begin{cases} h^{-1}(a) \simeq \mathrm{Prym}^0(\tilde{C}_a/C) \\ \bar{h}^{-1}(a) \simeq \mathrm{Prym}^0(\tilde{C}_a/C)/\Gamma \simeq (h^{-1}(a))^\vee \end{cases}$

Γ - action on cohomology

* $\Gamma \circlearrowright H^*(\mathcal{M}_{n,L})$ and $H^*(\overline{\mathcal{M}}_{n,d}) \simeq H^*(\mathcal{M}_{n,L})^{\Gamma}$.

* However : $H^*(\mathcal{M}_{n,L}) \neq H^*(\mathcal{M}_{n,L})^{\Gamma}$

(Compare with $H^*(\mathcal{N}_{n,L}) = H^*(\mathcal{N}_{n,L})^{\Gamma}$ (Harder–Narasimhan))

* Isotypical decomposition:

$$\boxed{H^*(\mathcal{M}_{n,L}, \Lambda) = \bigoplus_{\kappa \in \hat{\Gamma}} H^*(\mathcal{M}_{n,L}, \Lambda)_{\kappa}}$$

Γ - action and tautological classes

* We have

$$\begin{tikzcd} H^*(\check{\mathcal{M}}_{n,d}) \arrow[r, "res"] \arrow[dr] & H^*(\mathcal{M}_{n,L}) \arrow[d, phantom, "\rotatebox{90}{$\subsetneq$}"] & (*) \\ & H^*(\mathcal{M}_{n,L})^{\Gamma} & \end{tikzcd}$$

* Markman : $H^*(\check{\mathcal{M}}_{n,d})$ is generated by tautological classes, coming from Chern classes of $\mathcal{E}_{univ} \in Vec(\mathcal{M}_{n,d} \times C)$.

$\rightsquigarrow$ The strict inclusion (*) shows that $H^*(\mathcal{M}_{n,L})$ is not generated by the tautological classes from $\check{\mathcal{M}}_{n,d}$.

Weil pairing on Γ and cyclic covers

* $\Gamma = \mathrm{Pic}^\circ(C)[n]$ has a natural non-degenerate pairing:

$$\langle\ ,\ \rangle : \Gamma \times \Gamma \longrightarrow \mu_n$$

$$\rightsquigarrow \Gamma \xrightarrow{\sim} \hat{\Gamma}$$

* $\mathrm{Pic}^\circ(C)[n] \underset{AJ}{\simeq} H^1(C, \mathbb{Z}/n\mathbb{Z}) \simeq \mathrm{Hom}(\pi_1(C), \mathbb{Z}/n\mathbb{Z})$, so

$\gamma \in \Gamma$ gives rise to a cyclic cover $C_\gamma \xrightarrow{\pi} C$.

Fixed loci of Γ

* Let $\gamma \in \Gamma$. Write $M_\gamma := (M_{n,L})^\gamma$

* Since Γ is abelian, $\Gamma \circlearrowright M_\gamma$.

* $M_\gamma \xrightarrow{h_\gamma} A_\gamma := \mathrm{Im}(h|_{M_\gamma}) \xhookrightarrow{i_\gamma} A_n$

* $d_\gamma := \mathrm{codim}_{A_n}(A_\gamma) = \frac{1}{2}\,\mathrm{codim}_{M_{n,L}}(M_\gamma)$

Hausel - Thaddeus conjecture $\quad \Lambda = \mathbb{Q}(\zeta_n)$

Thm: (Groechenig-Wyss-Ziegler for Hodge numbers, p-adic integration; Maulik - Junliang Shen for Hodge structures, perverse sheaves + vanishing cycles)

we use this!

(i) $\gamma \in \Gamma \longleftrightarrow \kappa \in \hat{\Gamma}$.

$$H^*(\mathcal{M}_{n,L}, \Lambda)_\kappa \underset{\text{PHS}}{\simeq} H^{*-2d_\gamma}(\mathcal{M}_\gamma, \Lambda)_\kappa(-d_\gamma)$$

(ii) $H^*(\mathcal{M}_{n,L}, \Lambda) \simeq H^*_{orb}(\bar{\mathcal{M}}_{n,d}, \Lambda; \alpha)$

Rmk: (ii) $= \bigoplus_\kappa$ (i) by definition of twisted orbifold cohomology.
+ Hausel - Thaddeus def of the gerbe α.

Some related works:

* (Loeser - Wyss) HT conjecture holds in $\tilde{K}_0(CHM(k, \Lambda))$.
  motivic integration

* (Groechenig - Shiyu Shen, in preparation) HT holds for KU:
  Fourier-Mukai, vanishing cycles ...

$$KU(\mathcal{M}_{n,L}) \simeq KU(\bar{\mathcal{M}}_{n,d}, \alpha)$$

Conj (Semi - classical limit of geometric Langlands)
The Fourier-Mukai equivalences $D^b_{coh}(h_G^{-1}(a)) \simeq D^b_{coh}(h_{G^\vee}^{-1}(a))$ extend to a derived equivalence of $\mathcal{M}_G$ and $\mathcal{M}_{G^\vee}$ (rel $\mathcal{A}_G$)

Maulik - Shen proof strategy

Goal: Construct $\hat{\beta} : (\mathbb{R}R_* \Lambda)_\kappa \xrightarrow{\sim} i_{\gamma *}(\mathbb{R}R_{\gamma *}\Lambda)_\kappa(-d_\gamma)[-2d_\gamma]$ in $D^b_c(A_n)$

A) Work with D-twisted Higgs bundles

B) Construct a morphism in $D^b_c(A^D_n)$ ← usual top sheaves

$$\beta^D : (R^D_* \Lambda)_\kappa \longrightarrow i_{\gamma *}(R^D_{\gamma *}\Lambda)_\kappa(-d_\gamma)[-2d_\gamma]$$

C) Show that β^D is an iso using perverse sheaves.

D) Go back to usual Higgs bundles via vanishing cycles.

A) D-twisted Higgs bundles

- Let D divisor with either $\begin{cases} \mathcal{O}_C(D) \simeq \omega_C \\ D > 0,\ \deg(D) > 2g-2 \end{cases}$
- A D-Higgs bundle is (E, θ) with $\theta : E \longrightarrow E \otimes \mathcal{O}_C(D)$
- The theory then looks the same, except when $\deg > 2g-2$:
    - $\dim(A^D) > \frac{1}{2}\dim(M^D_{n,d})$
    - R^D has a "simpler topology".
    - $M^D_{n,d}, M^D_{n,L}$ are not symplectic anymore (Poisson, can be odd-dim)

B) Construction of β^D (deg D even $> 2g-2$)

- Recall :

$$\begin{tikzcd} \Gamma \circlearrowright \mathcal{M}_\gamma^D = \mathcal{M}_{n,L}^{D\gamma} \arrow[r, hook] \arrow[d, "h_\gamma^D"'] & \mathcal{M}_{n,L}^D \arrow[d, "h^D"] \arrow[loop right, "\Gamma"] \\ \mathcal{A}_\gamma^D \arrow[r, hook, "i_\gamma^D"] & \mathcal{A}_n^D = \bigoplus_{i=2}^n H^0(C, \mathcal{O}(iD)) \end{tikzcd}$$

Want morphism in $D_c^b(\mathcal{A}_n^D)$

- $\beta^D : (h_*^D \Lambda)_\kappa \longrightarrow i_{\gamma *}^D (h_{\gamma *}^D \Lambda)_\kappa(-d_\gamma^D)[-2d_\gamma^D] \in D_c^b(\mathcal{A}^D)$

Endoscopic correspondence (Ngo, Yun)

- β^D is constructed using
  - $*$ moduli of Higgs bundles on the cyclic cover C_γ
  - $*$ an explicit correspondence coming from geometry of generic Hitchin fibers on $\mathcal{A}_\gamma^D$.

- Fits into larger pattern of endoscopy for moduli of G-Higgs bundles $\rightsquigarrow$ Ngo's proof of fundamental lemma.

C) β^D isomorphism (deg D even > 2g-2)

- β^D generically isomorphism: explicit computation (Ngo-Yun)
- After taking perverse cohomology objects, both sides of β^D are intermediate extensions of local systems:
  * decomposition theorem
  * analysis of supports for R^D, R^D_π on the full Hitchin base

  (de Cataldo, Maulik-Shen; based on Ngo)
- Conclude β^D iso (Ngo's "perverse continuation principle")

D) Vanishing cycles Fix $p \in C$

* There is a ($\simeq$) quadratic form $\mu : \mathcal{A}_n^{D+p} \longrightarrow \mathbb{A}^1$ such that,

$$\mathcal{M}^D_{n,L} = \mathrm{Crit}(\mu \circ R^{D+p}) \xhookrightarrow{\iota} \mathcal{M}^{D+p}_{n,L}$$

* By applying $\varphi_\mu : D^b_c(\mathcal{A}_n^{D+p}) \longrightarrow D^b_c(\mathcal{A}_n^D)$ to β^{D+p}:

$$\hat{\beta}^D : (R^D_* \mathbb{C})_\kappa \xrightarrow{\sim} i_{\gamma *}(R^D_{\gamma *}\mathbb{C})_\kappa(-d_\gamma)[-2d_\gamma]$$

* Can go down to $D = K_C$ $\Rightarrow$ the proof is complete $\square$

III) Mixed motives, mixed motivic sheaves (Λ $\mathbb{Q}$-algebra)

* Voevodsky constructed a tensor triangulated Λ-linear category $DM(k, \Lambda)$ of mixed motives together with a motive tensor functor

$$M : Var_k \longrightarrow DM(k, \Lambda) \quad \left(M(X \times Y) = M(X) \otimes M(Y) \right)$$

* Idea: $(DM(k,\Lambda), M(-))$ universal pair satisfying:
  - Étale descent : $M(X)$ can be recovered from $M(Y)$ for $Y \to X$ étale covering.
    $C^\cdot(Y/X)$
  - $\mathbb{A}^1$-homotopy invariance : $M(X \times \mathbb{A}^1) \xrightarrow{\sim} M(X)$
  - $\mathbb{P}^1$-stability : $M_{red}(\mathbb{P}^1) =: \Lambda(1)[2]$ is $\otimes$-invertible

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Motivic sheaves

* S finite type $/k$ $\rightsquigarrow$ $DM(S, \Lambda)$ mixed motivic sheaves

* Six-functor formalism: (Ayoub) $f : T \longrightarrow S$

$$f^* : DM(S, \Lambda) \rightleftarrows DM(T, \Lambda) : f_*$$

$$f_! : DM(T, \Lambda) \rightleftarrows DM(S, \Lambda) : f^!$$

with similar properties to étale sheaves :
- proper base change
- Poincaré-Verdier duality

$\pi : X \to Spec(k)$

* $M(X) \simeq \pi_! \pi^! \Lambda$ homological motive.

$\pi_* \pi^* \Lambda$ coh.
$\pi_! \pi^* \Lambda$ coh. with comp.
$\pi_* \pi^! M$ BM

* " " BM -homolg

Motivic sheaves & the rest

$DM(-,\Lambda)$ relates both to cohomology and algebraic cycles:

* **Betti realisation functor** (Ayoub) : $k = \mathbb{C}$ (or $k \hookrightarrow \mathbb{C}$)

$$R_B : DM(S,\Lambda) \longrightarrow D(S^{an},\Lambda)$$

also sees MHS

which "commutes with the six operations".

* $CH_i(X) \otimes \Lambda \overset{(+)}{\simeq} \mathrm{Hom}\left(\Lambda(i)[2i], \pi_* \pi^! \Lambda(0)\right)$ (Voevodsky, Friedlander, Suslin)

$$\overset{X \text{ smooth}}{\simeq} \mathrm{Hom}\left(M(X), \Lambda(i)[2i]\right)$$

IV) Proof of main theorem

Thm (Hoskins-PL) In $DM(k,\Lambda)$, we have

(i) $\gamma \in \Gamma \longleftrightarrow \kappa \in \hat{\Gamma}$

$$M(\mathcal{M}_{n,L})_\kappa \simeq M(\mathcal{M}_\gamma)_\kappa (d_\gamma)[2d_\gamma]$$

(ii)

$$M(\mathcal{M}_{n,L}) \simeq M^{orb}(\bar{\mathcal{M}}_{n,d}, \alpha)$$

Cor: * Relation between (higher) Chow groups of $\mathcal{M}_{n,L}$ and $\mathcal{M}_\gamma$.

* Recovers Loeser-Wyss

Adapting the Maulik – Shen strategy

* **Goal:** $\hat{\beta}^{mot} : (R_* \Lambda)_\kappa \longrightarrow i_{\gamma*}(R_{\gamma*}\Lambda)_\kappa(-d_\gamma)[-2d_\gamma]$ in $DM(\mathcal{A}_n, \Lambda)$

  such that $\hat{\beta}^{mot}$ induces iso (i) (don't need $\hat{\beta}$ iso!)

* Steps A, B, D can be adapted directly to motivic sheaves:
  - A : identical
  - B : use same endoscopic correspondence and make it into a morphism in DM using (+)
  - D : relies on motivic vanishing cycles (Ayoub + ε to extend to stacks)

What about Step C?

- In an ideal motivic world, $DM(S, \Lambda)$ would admit a perverse motivic t-structure satisfying the decomposition theorem, with the same supports as in cohomology, and Step C would go through.

- At any rate, we pushforward to $\mathrm{Spec}(\mathbb{Q})$ and get morphism:

$$(v_* \hat{\beta}^{mot})^\vee : M(M_\gamma)_\kappa(d_\gamma)[2d_\gamma] \longrightarrow M(M_{n,L})_\kappa$$

whose realisation is an isomorphism by Maulik - Shen.

Conservativity of realisations

Conj: Let $k \subset \mathbb{C}$, Λ $\mathbb{Q}$-algebra. The Betti realisation

$$R_B : DM_c(k, \Lambda) \longrightarrow D^b_c(\Lambda)$$

is conservative, i.e detects isomorphisms.

$\Uparrow$ motivic t-str.

Thm: (Wildeshaus ; Kimura, Bondarko) $\mathrm{char}(k) = 0$.
This is true when restricting to the subcategory

$$DM^{ab}_c(k, \Lambda) := \langle M(x)^{(i)} \mid x \text{ curve}, i \in \mathbb{Z} \rangle^{\otimes, \mathrm{df}} \simeq \langle M(A) \mid A \text{ ab var} \rangle^{\mathrm{df}}$$

of abelian motives.

Abelian motives & moduli spaces on curves

Heuristic Motives of moduli spaces of bundles on a curve C tend to be abelian, often (but not always) in $\langle M(C) \rangle^{\otimes}$.
(more generally, motives of moduli of sheaves on X often in $\langle M(X) \rangle^{\otimes}$?)

Thm B (Hoskins - PL)

(i) $M(\check{\mathcal{M}}_{n,d}) \in \langle M(C) \rangle^{\otimes}$

(ii) $M(\mathcal{M}_{n,L})$ is abelian.

(iii) (H-PL-Fu) $M(\mathcal{M}_{2,L}) \notin \langle M(C) \rangle^{\otimes}$ for $C/_{\mathbb{C}}$ general $(g \geq 2)$

End of the proof:

$(v_* \hat{\beta}^{mot})^\vee : M(\mathcal{M}_\gamma)_K(d_\gamma)[2d_\gamma] \longrightarrow M(\mathcal{M}_{n,L})_K$

$\ast$ The RHS is abelian since $M(\mathcal{M}_{n,L})$ is. (Thm B.(ii))

$\ast$ The LHS is abelian because of the following

(also adapted from Maulik–Shen, based on Hausel–Pauly,

using finer study of Γ-action on $\mathcal{M}_{n,L}(\gamma)$ ($\leadsto$ moduli of Higgs bundles on C_γ.)

$$\begin{cases} M(\mathcal{M}_\gamma)_K \simeq M(\mathcal{M}_{r,L}(\gamma))^{G_\gamma \times \Gamma} \\ M(\mathcal{M}_{r,L}(\gamma))^\Gamma \text{ is a direct factor of } M(\check{\mathcal{M}}_{r,d}(C')) \text{ (Thm B.(i))} \end{cases}$$

$\ast$ Wildeshaus $\Rightarrow$ $(v_* \hat{\beta}^{mot})^\vee$ isomorphism ■

Relative version?

$\ast$ We expect $\hat{\beta}^{mot}$ itself to be an isomorphism in $DM(\mathcal{A})$.

$\ast$ It would suffice to show that the fibers of h all have abelian motives (in the GL_n and SL_n cases).

$\ast$ OK for : - nilpotent cone $h^{-1}(0)$ ($\Leftarrow$ total space via $\mathbb{G}_m$-action)

- fibers with reduced spectral curves.

Q Can one combine those two cases?