Macdonald polynomials

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Review of DAHA

Initial data

- Unlike in the previous talk, I will consistently use the notation from the survey paper "Cherednik Algebras, Macdonald polynomials and combinatorics" by Haiman, for ease of reference.
- Starting point: two reduced root data $(X, R, X^\vee, R^\vee)$ and $(Y, R', Y^\vee, R'^\vee)$. We assume they have the same Weyl group W_0, and fix $S \subset R$, $S' \subset R'$ and a bijection $s_i \leftrightarrow s_i'$ with $(W_0, S) \simeq (W_0, S')$.
- We assume for simplicity that the associated root systems are dual, so that we have a bijection $R' \simeq R^\vee$. The case when they are equal is also possible but slightly more complicated notationally. In particular, we have a canonical perfect W_0-equivariant pairing of root lattices $Q_0 \times Q_0' \to \mathbb{Z}$.
- We fix a W_0-extension of that pairing of the form

$$(-,-) : X \times Y \to \mathbb{Z}/m$$

for some integer $m \geq 1$.

Example: GL_n

We will illustrate some concepts in the case where both root data are equal to the self-dual GL_n root datum:

$$(\mathbb{Z}^n, \{e_i - e_j | 1 \leq i \neq j \leq n\}, \mathbb{Z}^n, \{e_i - e_j | 1 \leq i \neq j \leq n\})$$

with fixed choice of simple roots

$$S = \{\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n-1\}$$

and finite Weyl group $W_0 = S_n$. The pairing $X \times Y \to \mathbb{Z}$ is then the standard inner product on $\mathbb{Z}^n$, and $m = 1$ works.

Extended double affine Weyl groups

- Set $\tilde{X} = X \oplus \mathbb{Z}\delta/m$, $\tilde{Y} = Y \oplus Y \oplus \mathbb{Z}\delta'/m$. Extend the coroots by $\langle \delta, \alpha_i^\vee \rangle$. Set $\alpha_0 = \delta - \theta$ and let $\alpha_0^\vee$ be the extension of $-\theta^\vee$ with $\langle \delta, \alpha_0^\vee \rangle = 0$. This makes $\tilde{X}$ into an affine root system with simple roots $\tilde{S} = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ and simple coroots $(\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_n^\vee)$.
- GL_n: $\theta = e_1 - e_n$.
- Using $(-,-)$, we get an action of Y on $\tilde{X}$ (resp. X on $\tilde{Y}$) which extends to actions of extended affine Weyl groups $W_e = Y \rtimes W_0$ (resp. $W_e' = W_0 \ltimes X$) and gives rise to the extended double affine Weyl groups $W_e \ltimes \tilde{X}$ (resp. $\tilde{Y} \rtimes W_e'$).

Proposition

There is a canonical isomorphism $W_e \ltimes \tilde{X} \simeq \tilde{Y} \rtimes W_e'$ which is the identity on X, Y, W_0 and sends $q = x^\delta$ to $y^{-\delta}$.

The group formerly known as Ω

- There are also (non-extended) affine Weyl groups $W_a = Q_0' \rtimes W_0$ and $W_a' = W_0 \ltimes Q_0$. We have $W_a \subset W_e$ and $W_a' \subset W_e'$.
- Let $\Pi = W_e/W_a \simeq Y/Q_0'$ (abelian, not necc. finite). Then the pairing $(-,-)$ gives a canonical isomorphism $\Pi \simeq (X/Q_0)^* \simeq (W_e'/W_a')^* =: (\Pi')^*$.
- We have $W_e \simeq \Pi \ltimes W_a$ and $W_e' \simeq W_a' \ltimes \Pi^{'}$, and similarly for the extended double affine Weyl groups.
- For GL_n, the groups Π and Π' are free cyclic groups; Π' has generator π' acting on X (via W_e') as

$$\pi'(\lambda_1, \ldots, \lambda_n) = (\lambda_n + 1, \lambda_1, \ldots, \lambda_{n-1}).$$

Double affine braid groups

- Using the Coxeter group W_a, its extension W_e and the action on $\tilde{X}$, we can take inspiration from the Bernstein presentation of extended affine braid groups to define the (left) double affine braid group $B(W_e, \tilde{X}) := \Pi \ltimes B(W_a, \tilde{X})$ with $B(W_a, \tilde{X})$ the group generated by $B(W_a)$ and $\tilde{X}$ with the additional relations $(0 \leq i \leq n)$:
  - If $\langle \lambda, \alpha_i^\vee \rangle = 0$, then $x^\lambda T_i = T_i x^\lambda$.
  - If $\langle \lambda, \alpha_i^\vee \rangle = 1$, then $x^{s_i \lambda} = T_i x^\lambda T_i$.
- Similarly we have a (right) double affine braid group $B(\tilde{Y}, W_e') := B(\tilde{Y}, W_a') \rtimes \Pi'$ with $B(\tilde{Y}, W_a')$ generated by $B(W_a')$ and $\tilde{Y}$ together with the relations
  - If $\langle \lambda, \alpha_i^\vee \rangle = 0$, then $x^\lambda T_i = T_i x^\lambda$.
  - If $\langle \lambda, \alpha_i^\vee \rangle = 1$, then $x^{s_i \lambda} = T_i^{-1} x^\lambda T_i^{-1}$.

Cherednik duality theorem

Theorem

The isomorphism $W_e \ltimes \tilde{X} \simeq \tilde{Y} \rtimes W_e'$ *lifts to an isomorphism of double affine braid groups*

$$B(W_e, \tilde{X}) \simeq B(\tilde{Y}, W_e')$$

which is the identity on $\tilde{X}, \tilde{Y}$*, the braid group* $B(W_0)$ *and maps* $q = x^\delta$ *to* $q = y^{\delta'}$.

Double affine Hecke algebras

- We fix a commutative ground ring $\mathcal{A}$ and elements $u_i \in \mathcal{A}^\times$ for $0 \leq i \leq n$, with $u_i = u_j$ if α_i and α_j are in the same W-orbit.
- The (left) double affine Hecke algebra $\mathcal{H}(W_e, \tilde{X})$ is the quotient of the group algebra $\mathcal{A}B(W_e, \tilde{X})$ by the quadratic relations $(0 \leq i \leq n)$

$$(T_i - u_i)(T_i + u_i^{-1}) = 0.$$

- In other words, $\mathcal{H}(W_e, \tilde{X})$ is generated by elements $(x^\lambda)_{\lambda \in X}$, $\pi \in \Pi$, $T_0, \ldots, T_n$ and $x^\delta = q^{1/m}$ satisfying the relations of the double affine braid group and the quadratic relations. But recall that $W_e = Y \rtimes W_0$ so we also have elements $(y^\mu)_{y \in Y}$ in there!

Proposition (PBW property)

The elements $(y^\mu T_w x^\lambda)_{\mu \in Y, w \in W_0, \lambda \in X}$ form a basis of $\mathcal{H}(W_e, \tilde{X})$ as a free $\mathcal{A}[q^{\pm 1/m}]$-module.

Duality

The (right) double affine Hecke algebra $\mathcal{H}(\tilde{Y}, W'_e)$ is the quotient of the group algebra $\mathcal{A}B(W_e, \tilde{X})$ by the quadratic relations $(0 \leq i \leq n)$

$$(T_i - u_i)(T_i + u_i^{-1}) = 0.$$

(There is a reindexing of the parameters u_i in some cases which I will not explain.)

Theorem

There is an isomorphism $\mathcal{H}(W_e, \tilde{X}) \simeq \mathcal{H}(\tilde{Y}, W'_e)$ which is an isomorphism on all the generators $X, Y, q, T_i, T_0, T'_0, \Pi, \Pi'$.

Polynomial representation

- The (extended) affine Hecke algebra $\Pi \cdot \mathcal{H}(W_a)$ has a representation on a one-dimensional $\mathcal{A}$ module $\mathcal{A} \cdot e$ with Π acting trivially and $T_i e := u_i e$.
- The algebra $\Pi \cdot \mathcal{H}(W_a)$ is naturally a sub-algebra of the DAHA. The induced representation $\mathrm{Ind}_{\Pi \cdot \mathcal{H}(W_a}^{\mathcal{H}(W_e, \tilde{X}}(\mathcal{A} \cdot e)$ is called the polynomial representation. By the PBW property, its underlying module is $\mathcal{A}X$, with X acting by left multiplication, Π acting via its action on X, and $T_0, \ldots, T_n$ acting as the operators

$$T_i = u_i s_i + \frac{u_i - u_i^{-1}}{1 - x^{\alpha_i}}(1 - s_i).$$

Macdonald polynomials: definition

The space of polynomials

- Same setup, same notations, but the definition of Macdonald polynomials only depends on the "X side". The "Y side" is used to prove the existence of Macdonald polynomials and study their property.
- We write $\mathbb{Q}(t) := \mathbb{Q}(u_i, 0 \leq 1 \leq n)$, with the convention that $t_i = u_i^2$.
- The group algebra $\mathbb{Q}(t)\tilde{X}$ is a ring of Laurent polynomials over the field $\mathbb{Q}(t)$. We put $q = x^{\delta}$, so that $\mathbb{Q}(t)\tilde{X} = \mathbb{Q}(t)[q^{\pm 1/m}]X$. This lies in $\mathbb{Q}(t, q^{\pm 1/m})X$, which we write as $\mathbb{Q}(t, q)$ by a small abuse of notation.

An orthogonality kernel

- Let $\mathbb{Q}(q,t)X^\wedge$ denote the $\mathbb{Q}(q,t)$-vector space of possibly infinite linear combinations of elements of X. It is a module over $\mathbb{Q}(q,t)X$. For $f \in \mathbb{Q}(q,t)X^\wedge$, we write $[x^\lambda]f$ for the coefficient of x^λ.
- Let $\overline{(-)}$ be the involution on $\mathbb{Q}(q,t)$ and $\mathbb{Q}(q,t)X$ defined by

$$\bar{u}_i = u_i^{-1},\ \bar{q} = q^{-1},\ \bar{x^\lambda} = x^{-\lambda}$$

Proposition

There exists a unique element $\Delta_0 \in \mathbb{Q}(q,t)Q_0^\wedge \subset \mathbb{Q}(q,t)X^\wedge$ *with* $\bar{\Delta}_0 = \Delta_0$, $[1]\Delta_0 = 1$ *and for every* $0 \le i \le n$,

$$s_i(\Delta_0) = \frac{1 - t_i x^{\alpha_i}}{t_i - x^{\alpha_i}} \Delta_0.$$

The idea is to define an infinite product $\Delta := \prod_{\alpha \in \tilde{R}^+} \frac{1-x^\alpha}{1-t_\alpha x^\alpha}$ and to show $\Delta_0 = \Delta/([1]\Delta)$ works.

Cherednik inner product

- The Cherednik inner product on $\mathbb{Q}(q,t)X$ is defined as

$$\langle f, g\rangle_0 := [1](f\bar{g}\Delta_0)$$

  It is linear in f and $(\bar{-})$-hermitian.
- Over $\mathbb{C}$ and with some assumptions on the parameters, this can be described analytically as integration over the compact torus $T_u := \mathrm{Hom}(X, S^1)$ with respect to a certain meromorphic kernel defined by a similar infinite product.

Macdonald polynomials

There is a natural partial order $\leq$ on X, which on X_+ is simply given by $\lambda \leq \mu$ iff $\mu - \lambda$ can be written as a sum of positive roots.

Theorem

There is a unique basis $(E_\lambda)_{\lambda \in X}$ of $\mathbb{Q}(q,t)X$, the (non-symmetric) Macdonald polynomials, satisfying

- $\langle E_\lambda, E_\nu \rangle_0 = 0$ *for* $\lambda \neq \nu$.
- $E_\lambda = x^\lambda + \sum_{\nu < \lambda} c_{\lambda\nu} x^\nu$.

Macdonald polynomials: construction using DAHA

DAHA action

- We now use the full initial datum, including the Y side, to introduce the DAHA $\mathcal{H} := \mathcal{H}(W_e, \tilde{X})$ with $\mathcal{A} = \mathbb{Q}(t)$.
- We identify $\mathbb{Q}(q,t)X$ with the space of the polynomial representation of $\mathcal{H}$.

Proposition

Let $0 \leq i \leq n$. The operator T_i acting on $\mathbb{Q}(q,t)X$ is unitary with respect to Cherednik's inner product.

By the quadratic relation and the fact that $\bar{u}_i = u_i^{-1}$, this is equivalent to the fact that $T_i - u_i$ is self-adjoint, which is a direct computation because of the defining property of Δ_0 and the equation

$$T_i = u_i s_i + \frac{u_i - u_i^{-1}}{1 - x^{\alpha_i}}(1 - s_i).$$

Cherednik operators

- The new structure afforded by the DAHA action is the action of the elements y^{μ}. The corresponding operators on $\mathbb{Q}(q,t)$ are the Cherednik operators.
- For notation convenience, introduce "formal logarithms" k_i for $i \neq 0$ with $q^{k_i} = u_i$, and extend to the whole of R by the action of W_0. Put $\rho'^{\vee} = \sum_{\alpha \in R_+} k_\alpha \alpha'^{\vee}$ (where $\alpha' \in R'$ is such that $s_\alpha = s_{\alpha'}$).

Proposition

The Cherednik operators satisfy

$$y^{\mu}(x^{\lambda}) = q^{-(\lambda,\mu)+\langle \mu, w_\lambda(\rho'^{\vee})\rangle} x^{\lambda} + \sum_{\nu < \lambda} b_{\lambda\nu} x^{\nu}$$

with w_λ the minimal representative of $x^{\lambda} W_0$ in the Bruhat order of W'_e.

The proof reduces immediately to $\mu \in Y^{+}$, hence $y^{\mu} = T_{y^{\mu}}$, the case $y^{\mu} = s_i$ for $0 \leq i \leq n$ is direct, and the general case follows by induction on the length.

End of the proof

- The T_i are unitary, and the y^μ are composites of T_i's and T_i^{-1}'s, so they are unitary as well.
- The y^μ's commute by construction.
- The y^μ's preserve the subspace $(\mathbb{Q}(q,t)Q_0)X$ and act on it as lower triangular operators with distinct eigenvalues.
- All of these properties imply that for a fixed $\lambda \in X$ they admit joint eigenfunctions E_λ with eigenvalues $q^{-(\lambda,\mu)+\langle \mu, w_\lambda(\rho'^\vee)\rangle}$, normalised with $[x^\lambda]E_\lambda = 1$, which are mutually orthogonal. These are precisely the Macdonald polynomials.

Intertwiner relations

Intertwiner relations

- The existence of Macdonald polynomials was known before the work of Cherednik, although the DAHA proof is quite elegant. The DAHA approach gives much more, however.
- The general commutation relations

$$T_i x^{\lambda} - x^{s_i(\lambda)} T_i = \frac{u_i - u_i^{-1}}{1 - x^{\alpha_i}} (x^{\lambda} - x^{s_i(\lambda)})$$

  together with the duality theorem to get similar formulas for the y^{μ}'s can be used to relate Macdonald polynomials for different λ's. The precise formulas are complicated and I will not reproduce them here. These intertwiner relations are key to proving properties of Macdonald polynomials by "induction on λ".

Intertwiner relations for GL_n

The Macdonald polynomials for GL_n are parametrized by $X = \mathbb{Z}^n$. The intertwiner relations for GL_n are also known as Knop's reccurence. We have $E_{(0,\ldots,0)} = 1$ as base case of the induction, and two other formulas which together determine all the E_λ's:

$$E_{(\lambda_n+1,\lambda_1,\ldots,\lambda_{n-1})} = q^{\lambda_n} x_1 E_\lambda(x_2,\ldots,x_n,x_1/q).$$

$$E_{s_i(\lambda)} = \left(u_i T_i + \frac{1-t}{1-q^{\lambda_i-\lambda_{i+1}} t^{\bar\lambda_i-\bar\lambda_{i+1}}} \right) E_\lambda$$

with $(\bar\lambda_i)$ the permutation of $(1,\ldots,n)$ such that $\bar\lambda_i > \bar\lambda_j$ iff $\lambda_i > \lambda_j$.

Symmetric Macdonald polynomials

- Let $\lambda \in X^+$ and $V_\lambda = \mathbb{Q}(q,t)\{E_\nu, \nu \in W_0\lambda\}$. The intertwiner relations imply that V_λ is a $\mathcal{H}(W_0)$-submodule of $\mathbb{Q}(q,t)X$. Applying the symmetriser idempotent of that finite Hecke algebra, we see that there is a unique W_0-invariant element $P_\lambda \in V_\lambda$ such that $[x^\lambda]P_\lambda = 1$, the symmetric Macdonald polynomial.
- The P_λ's are orthogonal wrt Cherednik's inner product, and eigenfunctions of the W_0-invariant operators coming from $(\mathbb{Q}(q,t)Y)^{W_0}$.
- They can also be characterized using a symmetrisation $\langle\rangle'$ of Cherednik's inner product, and this is how they were first introduced by Macdonald.
- In the GL_n-case, they are symmetric functions which generalise Schur functions, Hall-Littlewood and Jack polynomials...

Norm and evaluation formulas

The intertwiner relations were used by Cherednik to prove formulas for $\langle E_\lambda, E_\lambda \rangle$ and on the values of E_λ. By symmetrisation, one then gets the original conjectures of Macdonald concerning the P_λ's. For instance, assuming $t = t_i = q^k$ for some $k \geq 1$, one has

$$\langle P_\lambda, P_\lambda \rangle' = \prod_{\alpha \in \tilde{R}^+} \prod_{i=1}^{k} \frac{1 - q^{2(\alpha^\vee, \lambda + k\rho + 2i)}}{1 - q^{2(\alpha^\vee, \lambda + k\rho - 2i)}}.$$