Motives of moduli of bundles on curves

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Plan:

I) Moduli of bundles on curves

II) Mixed motives & motivic sheaves

III) Motive of Bun

IV) Motivic mirror symmetry

I) Moduli of bundles on curves

$$\begin{cases} C & \text{smooth projective geometrically connected curve over field } k. \\ G & \text{reductive algebraic group } / k \quad (\text{today, } G = GL_n, SL_n, PGL_n) \end{cases}$$

⇝ many interesting moduli spaces & stacks of "G-bundles".

- principal G-bundles
- G-Higgs bundles
- G-bundles with connection
- Parabolic variants

...

Moduli of vector bundles

* Two basic objects $\begin{cases} \text{moduli } \underline{\underline{\text{stack}}} \ \mathrm{Bun} \text{ of all vector bundles} \\ \text{moduli } \underline{\underline{\text{space}}} \ \mathcal{N} \text{ of semistable vector bundles} \end{cases}$

* Discrete invariants = (rank, degree) $\in \mathbb{N} \times \mathbb{Z} = \pi_0(\mathrm{Bun})$

$$\mathrm{Bun}_{n,d}(C)(T) := \left(E \underset{\text{rk } n, \deg d}{\xrightarrow{\text{v.b.}}} C \times T \text{, bundle isomorphisms} \right) \leftarrow \text{groupoid}$$

smooth Artin stack of dimension $n^2(g-1)$

Def: E is (semi)stable if $\forall \underset{\neq 0}{E'} \subsetneq E$, $\frac{\deg E'}{\operatorname{rk} E'} \overset{(\leq)}{<} \frac{\deg E}{\operatorname{rk} E} =: \underset{\substack{\nearrow \\ \text{slope}}}{\mu}(E)$

- $\mathcal{N}_{n,d}$ "coarse" (adequate) moduli space of semistable vb of rank n deg d :

  projective variety of dimension $n^2(g-1)+1$

- $\mathcal{N}^s_{n,d} \overset{\circ}{\hookrightarrow} \mathcal{N}_{n,d}$ moduli space of stable ————

  smooth, $\mathcal{N}^s_{n,d} = \mathcal{N}_{n,d}$ when $(n,d)=1$.

- Other basic object: $\mathrm{Sym}^d(C)$ (moduli of effective divisors)

Harder - Narasimhan filtration

- Every vector bundle $E \in \mathrm{Bun}(C)$ has a unique Harder-Narasimhan filtration $\{0\} \subseteq E_1 \subseteq E_2 \subseteq \dots \subseteq E$

  with
$$\begin{cases} E_i/E_{i-1} \text{ semistable of slope } \tau_i = \frac{d_i}{n_i} \\ \tau_1 > \tau_2 > \cdots \end{cases}$$

- $\tau = \{\tau_i\}$ Harder - Narasimhan type of E

Harder - Narasimhan stratification

$$\mathrm{Bun}_{n,d} = \mathrm{Bun}^{ss}_{n,d} \sqcup \coprod_{\tau \neq \{\frac{d}{n}\}} \mathrm{Bun}^{\tau}_{n,d}$$ locally closed stratification

with
$$\left\{ \begin{array}{ll} \mathrm{Bun}^{ss}_{n,d} \longrightarrow \mathcal{N}^{ss}_{n,d} & \text{good / adequate moduli space} \\ (\mathbb{G}_m\text{-gerbe for } (n,d) = 1) & \\ \mathrm{Bun}^{\tau}_{n,d} \xrightarrow{gr} \prod_i \mathrm{Bun}^{ss}_{n_i,d_i} & \text{v.bundle stack.} \end{array} \right.$$

$(GL_n$-$)$ Higgs bundles (Hitchin)

- Pair (E, θ) with

$$\left\{ \begin{array}{l} E \quad \text{vector bundle of rank } n, \text{ degree } d \text{ on } C \\ \theta : E \xrightarrow{\mathcal{O}_C} E \otimes \omega_C \end{array} \right.$$ Higgs field

- Appears via cotangent of Bun :

$$T^*_E \mathrm{Bun}_{n,d} \overset{\substack{\text{defn.} \\ \text{theory}}}{\underset{\substack{\uparrow \\ \text{neglect automorphisms}}}{\simeq}} \mathrm{Ext}^1(E,E)^* \overset{\substack{\text{Serre} \\ \text{duality}}}{\simeq} \mathrm{Hom}(E, E \otimes \omega_C)$$

Moduli space of Higgs bundles (Hitchin, Simpson, Nitsure, ...)

* $\mathcal{M}_{n,d}$ moduli space of semi stable Higgs bundles:

- quasiprojective, of dimension $2(n^2(g-1)+1)$
- smooth algebraic symplectic variety $\supset T^*\mathcal{N}_{n,d}$ for $(n,d)=1$

* Hitchin fibration: proper, Lagrangian morphism

$$\begin{aligned} h : \mathcal{M}_{n,d} &\longrightarrow \mathcal{A}_n := \bigoplus_{i=1}^{n} H^0(C, \omega_C^{\otimes i}) \\ (E,\theta) &\longmapsto (\mathrm{Tr}(\Lambda^i \theta)) \end{aligned}$$

to the Hitchin base $\mathcal{A}_n$.

global analogue of

$$\mathfrak{g} \longrightarrow \mathfrak{t}/W$$

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* Spectral curve

$$\begin{tikzcd} \mathcal{C}_a := P_a^{-1}(0) \arrow[r, phantom, "\subseteq"] \arrow[rrr, bend right, "n:1"'] & T^*C \arrow[rr] & & C \end{tikzcd}$$

$\left(P_a = \sum a_i t^{n-i}\right)$

$\mathcal{A}_n^{sm} \subset \mathcal{A}_n$ open locus where $\mathcal{C}_a$ is smooth.

$$\mathcal{M}_{n,d}^{sm} \simeq \mathrm{Pic}^d(\mathcal{C}^{sm}/\mathcal{A}_n^{sm}) \circlearrowleft \mathrm{Jac}(\mathcal{C}^{sm}/\mathcal{A}^{sm})$$

$\rightsquigarrow$ h is a completely integrable Hamiltonian system

* Other fibers of h "more and more singular", culminating in nilpotent cone $h^{-1}(0)$.

* **Hitchin scaling action** :

$$\mathbb{G}_m \circlearrowright \mathcal{M}_{n,d} \quad , \text{"} t \cdot (E,\theta) = (E, t\theta) \text{"}$$

Semi projective : - $(\mathcal{M}_{n,d})^{\mathbb{G}_m}$ is projective / h

- $\lim_{t \to 0} t \cdot (E,\theta)$ always exists. $\left(\in h^{-1}(0) \right)$

Białynicki - Birula $(n,d) = 1$

$$\begin{cases} \mathcal{M}_{n,d} = \displaystyle\coprod_{[W_i] \in \pi_0(\mathcal{M}_{n,d}^{\mathbb{G}_m})} W_i^+ \\ W_i^+ \longrightarrow W_i \quad \text{affine space bundle} \end{cases}$$

Moduli of Higgs bundles : applications

- **Hyperkähler** counterpart to $\mathcal{N}_{n,d}$ (Hitchin, ···

Gauge theory for "self-dual Yang-Mills functional" Nietzke-Gaiotto-Moore)

- **Non-abelian Hodge theory** : relation with Betti/de Rham local systems,

$$\mathcal{M}_{n,d} \overset{C^\infty}{\simeq} \underset{\substack{\uparrow \\ \text{twisted vb} \\ \text{with connect}^\circ}}{\mathcal{M}_{n,d}^{dR}} \underset{RH}{\overset{Hol}{\simeq}} \underset{\substack{\uparrow \\ \text{twisted} \\ \text{local systems}}}{\mathcal{M}_{n,d}^{B}}$$

(Simpson, Corlette, ···)

P = W conjecture (Simpson, Corlette, Hausel, de Cataldo, Migliorini ...)

More relevant for this talk:

- Semiclassical limit of the geometric Langlands program (Hausel-Thaddeus, Donagi-Pantev, Kapustin-Witten, ...)

- Global and affine Springer theory and Langlands for local and global function fields; fundamental lemmas (Ngo, Yun, ..., Bezrukavnikov-Boixeda Alvarez-Mac Breen-Yun)

SL_n- and PGL_n-Higgs bundles

$(n, d) = 1$ until the end!

* Fix $L \in \mathrm{Pic}^d(C)$. A (twisted) SL_n-Higgs bundle is a Higgs bundle with
$$\begin{cases} \det(E) \simeq L & \text{in } \mathrm{Pic}^d(C) \\ \mathrm{Tr}(\theta) = 0 & \text{in } H^0(C, \omega_C) \end{cases}$$

* A PGL_n-Higgs bundle of degree d is ... a special case of the general definition of G-Higgs bundles!

$\left(\mathcal{E}/_C \text{ principal } G\text{-bundle} + \theta \in H^0\left(C, \mathrm{Ad}(\mathcal{E}) \otimes \omega_C\right)\right)$

Moduli of SL_n- and PGL_n-Higgs bundles

$$\begin{cases} \mathcal{M}_{n,L} \hookrightarrow \mathcal{M}_{n,d} \quad \text{smooth of codimension } 2g = \begin{matrix} \dim \operatorname{Pic}^d \\ + \\ \dim H^0(C,\omega_C) \end{matrix} \\ h : \mathcal{M}_{n,L} \longrightarrow \overline{\mathcal{A}}_n := \bigoplus_{i=2}^{n} H^0(C, \omega_C^{\otimes i}) \quad \text{proper, Lagrangian} \end{cases}$$

* $\operatorname{Jac}(C)$ acts on $\mathcal{M}_{n,d}$ by tensoring

$\cup$

$\Gamma := \operatorname{Jac}(C)[n]$ " " $\mathcal{M}_{n,L}$ " " "

* $\overline{\mathcal{M}}_{n,d} := \left[\mathcal{M}_{n,L} / \Gamma \right] \xrightarrow{\overline{h}} \overline{\mathcal{A}}_n$ ← same Hitchin base !

Alternatively:

$\overline{\mathcal{M}}_{n,d} \simeq \left[\mathcal{M}_{n,d} / T^*\operatorname{Jac} \right]$

Mirror symmetry for Higgs bundles

$G, G^\vee$ Langlands dual reductive groups $\Longrightarrow$

$$\begin{tikzcd} \mathcal{M}_G \arrow[dr, "h_G"'] & & \mathcal{M}_{G^\vee} \arrow[dl, "h_G"] \\ & \mathcal{A}_G \simeq \mathcal{A}_{G^\vee} & \end{tikzcd}$$

Thm (Hausel-Thaddeus for $G = SL_n$; Donagi-Pantev in general)

- For generic $a \in \mathcal{A}_G$ the Hitchin fibers $h_G^{-1}(a)$ $h_{G^\vee}^{-1}(a)$ are dual abelian varieties. (HyperKähler SYZ)
- For $G = SL_n$: $\begin{cases} h^{-1}(a) \simeq \operatorname{Prym}^0(\mathcal{C}_a/C) \\ \overline{h}^{-1}(a) \simeq \operatorname{Prym}^0(\mathcal{C}_a/C)/\Gamma \simeq (h^{-1}(a))^\vee \end{cases}$

Cor:

$$D^b_{coh}\left(h_G^{-1}(a)\right) \underset{FM(\mathcal{P})}{\overset{(\text{Mukai})}{\simeq}} D^b_{coh}\left(h_{G^\vee}^{-1}(a)\right)$$

"Conj": (Semiclassical limit of GLC ; Donagi-Pantev)

This extends to a derived equivalence

$$D^b_{coh}(\mathcal{M}_G) \overset{?}{\simeq} D^b_{coh}(\mathcal{M}_{G^\vee}) \quad \text{rel } \mathfrak{A}_{G^{(\vee)}}$$

Rmk: - Imprecise statement! Need stacks, gerbes; known over $\mathfrak{A}^{sm}_G$.

- Also expect corrections, similar to $QCoh \rightsquigarrow IndCoh_{\mathcal{N}}$ in GL?

Heuristic link with Geometric Langlands

"Conj" (naive GLC, Beilinson-Drinfeld)

$$D\text{-Mod}(Bun_G) \overset{?}{\simeq} QCoh(\mathcal{M}^{dR}_G)$$

$$\begin{tikzcd}
D\text{-Mod}(Bun_G) \arrow[r, rightsquigarrow, "\hbar \to 0"] \arrow[d, phantom, "\wr\wr\ ?"] & gr\,\mathcal{D}\text{-Mod}(Bun_G) \simeq QCoh(\mathcal{H}iggs_G) \arrow[d, phantom, "\wr\wr\ ?"] & \\
QCoh(\mathcal{M}^{dR}_{G^\vee}) \arrow[r, shift right, "\text{ext from } \mathbb{G}_m"'] & QCoh(\mathcal{M}^{Hdg}_{G^\vee}) \arrow[l, hook', shift right, "\lambda \neq 0"'] \arrow[r, "\lambda \to 0"] & QCoh(\mathcal{H}iggs_{G^\vee}) \\
 & \lambda\text{-connections} \arrow[u] &
\end{tikzcd}$$

Kapustin-Witten: "GLC itself can be recovered from $Br(\mathcal{M}_G) \overset{S}{\simeq} Br(\mathcal{M}_{G^\vee})$"

Recap :

$$\begin{tikzcd}
\boxed{\mathrm{Bun}_{n,d}} \arrow[r, leftrightarrow, "\circ" marking] \arrow[d, hook] & \mathrm{Bun}^{ss}_{n,d} \arrow[r] & \mathcal{N}_{n,d} \arrow[d, hook] & & \\
T^*\mathrm{Bun}_{n,d} \arrow[d, phantom, "\cong"] & & T^*\mathcal{N}_{n,d} \arrow[d, hook, "\circ" marking] & & \\
\mathrm{Higgs}_{n,d} \arrow[r, leftrightarrow, "\circ" marking] & \mathrm{Higgs}^{ss}_{n,d} \arrow[r] & \mathcal{M}_{n,d} \arrow[r, "R"] \arrow[d, hook] & \mathcal{A}_n & \\
& & \mathcal{M}_{n,L} \arrow[r, "R"'] & \bar{\mathcal{A}}_n \arrow[u, hook] & \bar{\mathcal{M}}_{n,d} \arrow[l, hook', "\bar{R}"'] \\
& & \Gamma \arrow[u, phantom, "\circlearrowleft"] & &
\end{tikzcd}$$

$\square$ = rest of the talk

Questions ?

I) Mixed motives, mixed motivic sheaves (Λ $\mathbb{Q}$-algebra)

* Voevodsky constructed a tensor triangulated Λ-linear category $DM(k,\Lambda)$ of mixed motives together with a motive tensor functor

$$M : Var_k \longrightarrow DM(k,\Lambda) \qquad \big(M(X \times Y) = M(X) \otimes M(Y) \big)$$

* Partial prehistory:

- Grothendieck
- Deligne (yoga of weights, mixed Hodge theory, ...)
- Beilinson (conjectural picture, inspired by conjectures on L-funct°)

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* Idea: $(DM(k,\Lambda), M(-))$ ↝ universal pair satisfying:

- Étale descent : $M(X)$ can be recovered from $M(\check{C}(Y/X))$, $\begin{matrix} Y \\ \downarrow \\ X \end{matrix}$ étale cov.
- $\mathbb{A}^1$-homotopy invariance : $M(X \times \mathbb{A}^1) \xrightarrow{\sim} M(X)$
- $\mathbb{P}^1$-stability : $M_{red}(\mathbb{P}^1) =: \Lambda(1)[2]$ is $\otimes$-invertible

* Voevodsky's insight : these standard properties of Weil cohomology theories "imply everything else".

* Rmk: treat $DM(k,\Lambda)$ as a presentable Λ-linear stable symmetric monoidal ∞-category; then

$$\begin{tikzcd} \text{Idea} \arrow[d] \\ \text{Thm-Def} \end{tikzcd}$$

!

Motivic sheaves

* S finite type $/k$ $\rightsquigarrow$ $DM(S,\Lambda)$ mixed motivic sheaves

* Six-functor formalism: (Ayoub) $f : T \longrightarrow S$

$$f^* : DM(S,\Lambda) \rightleftarrows DM(T,\Lambda) : f_*$$

$$f_! : DM(T,\Lambda) \rightleftarrows DM(S,\Lambda) : f^!$$

with similar properties to étale sheaves (+ vanishing cycles

* $\pi : X \to \mathrm{Spec}(k)$

$$M(X) \simeq \pi_! \pi^! \Lambda$$ homological motive.

Motivic sheaves vs other invariants

$DM(-,\Lambda)$ relates both to cohomology and algebraic cycles :

* Betti realisation functor : $k = \mathbb{C}$ (or $k \hookrightarrow \mathbb{C}$)
(Ayoub)

$$R_B : DM(S,\Lambda) \longrightarrow D(S^{an},\Lambda)$$

also sees MHS

which “commutes with the six operations”.

* $$CH_i(X) \otimes \Lambda \overset{(+)}{\simeq} \mathrm{Hom}\left(\Lambda(i)[2i], \pi_* \pi^! \Lambda(0)\right)$$ (Voevodsky, Friedlander, Suslin)

$$\overset{X \text{ smooth}}{\simeq} \mathrm{Hom}\left(M(X), \Lambda(i)[2i]\right)$$

Motives of stacks

- Using étale => smooth descent, can make sense of $M(\mathfrak{X}) \in DM(k,\Lambda)$ for Artin stack $\mathfrak{X}/k$.
- For quotient stacks, can compute « à la Totaro »:

$$M([X/G]) = \operatorname{colim}_n M\left(\frac{X \times (V^n)^{\text{free}}}{G}\right) \quad V \text{ faithful } G\text{-rep.}$$

- Also robust theory of motives on stacks (e.g. Richarz - Scholbach)

Motivic t-structure conjecture

- $DM(k,\Lambda)$ admits many variants, most notably $\begin{cases} DM^{Nis}(k,\mathbb{Z}) \\ SH(k) \end{cases}$

$\rightsquigarrow$ motivic homotopy theory.

- However $DM(k,\Lambda)$ is conjecturally particularly simple.

Conj: There exists a t-structure on $DM(k,\Lambda)$ such that realisat° functors are t-exact.

This implies most open conjectures on algebraic cycles...

Conservativity of realisations

Conj: Let $k \subset \mathbb{C}$, Λ $\mathbb{Q}$-algebra. The Betti realisation

$$R_B : DM_c(k,\Lambda) \longrightarrow D^b_c(\Lambda)$$

is conservative, i.e detects isomorphisms.

$\Uparrow$ motivic t-str.

Thm: (Wildeshaus ; Kimura, Bondarko) char$(k) = 0$.

This is true when restricting to the subcategory

$$DM^{ab}_c(k,\Lambda) := \left\langle M(X)(i) \;\middle|\; \begin{matrix} X \text{ curve} \\ i \in \mathbb{Z} \end{matrix} \right\rangle^{\otimes, df} \simeq \left\langle M(A) \mid A \text{ ab var} \right\rangle^{df}$$

of abelian motives.

Questions ?

III) Motive of Bun

Classical results on cohomology of $\mathrm{Bun}_{n,d}$, $\mathcal{N}_{n,d}$:

- Harder-Narasimhan :

  point count $/\mathbb{F}_q$ + Weil conj + HN recursion

  $\Rightarrow$ recursive formulas for Betti numbers of $\mathcal{N}_{n,d}$.

- Atiyah-Bott ($k = \mathbb{C}$)

  Gauge theory $\rightsquigarrow$ formula for $H^*(\mathrm{Bun}_G(\mathbb{C}))$

- Bifet-Ghione-Letizia :

  Algebro-Geometric proof of HN / AB for GL_n :

$$\mathrm{Div}(\overset{\text{eff. divisor}}{D}) = \left\{ E \subseteq \mathcal{O}_C(D)^{\oplus n} \;\middle|\; \begin{matrix} \mathrm{rk}\, E = n \\ \deg E = d \end{matrix} \right\}$$

smooth proj. variety of matrix divisors

$$H^*(\mathrm{Div}(D)) \text{ accessible via } \begin{cases} \text{BB dec. (BGL)} \\ \text{Flag Div } \left(\begin{matrix}\text{Heinloth} \\ \text{HPL}\end{matrix}\right) \end{cases}$$

$$\mathrm{Div} := \underset{\text{"}D \to \infty\text{"}}{\mathrm{colim}}\ \mathrm{Div}(D) \longrightarrow \mathrm{Bun}$$

$$E \subseteq \mathcal{O}_C(D)^{\oplus n} \longmapsto E$$

BGL : "Div $\longrightarrow$ Bun is an infinite rank v.bundle."

- **Gaitsgory - Lurie** :

Computation of $H^*_e(\mathrm{Bun}_G)$ via Beilinson - Drinfeld affine Grassmannian and factorisation homology.

Thm **(HPL)** Assume $C(k) \neq \emptyset$. Then

$$M(\mathrm{Bun}_{n,d}(C)) \simeq M(\mathrm{Jac}(C)) \otimes M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{n-1} \bigoplus_{j \geq 0} \mathrm{Sym}\ M(C)\{ij\}$$

Sketch: Following BGL, using an approximation of Bun by open substack which are quotient stacks and computing "à la Totaro",

$$M(\mathrm{Bun}_{n,d}) \simeq \underset{\ell \to \infty}{\mathrm{colim}}\ M(\mathrm{Div}(\ell D_0))$$

Pb: We understand each term but not transition maps.

(Functoriality of BB decomposit° along $\mathbb{G}_m$-eq. maps is not so easy)

Enough for formula in $\hat{K}_0(\mathrm{Var}_k)$ (Behrend-Dhillon) but not for $\mathrm{DM}(k, \mathbb{Q})$.

Solution: Fix $x \in C(k) \neq \emptyset$.

$$\mathrm{FDiv}(\ell x) := \{ E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{n\ell - d} = \mathcal{O}(\ell x)^{\oplus n} \}$$

$\big\downarrow$ ⟵ small morphism (Laumon, case $C = \mathbb{A}^1$ related to Grothendieck-Springer resolution)

$\mathrm{Div}(\ell x)$

$\Downarrow$ ⤳ de Cataldo-Migliorini

$$\begin{cases} M(\mathrm{Div}(\ell x)) \simeq M(\mathrm{FDiv}(\ell x))^{S_N} \\ \mathrm{FDiv}(\ell x) \text{ it° projective bundle with symmetric powers of } C, \text{ explicit transit° maps.} \end{cases} \quad \square$$

Cor | $M(\mathrm{Bun}_{n,d}) \in \langle\!\langle M(C) \rangle\!\rangle^{\otimes}$

Questions?

IV) Motivic mirror symmetry ($k = \bar{k}$, $\mathrm{char}(k) = 0$)

Recap :

$$\begin{tikzcd}
\mathcal{M}_{n,L} \arrow[loop right, "\Gamma"] \arrow[dr, "h"'] & & \overline{\mathcal{M}}_{n,d} = \left[\mathcal{M}_{n,L} / \Gamma \right] \arrow[dl, "\bar{h}"] \\
& \overline{\mathcal{A}}_n &
\end{tikzcd}$$

Conj | $$D^b_{coh}(\mathcal{M}_{n,L}) \overset{?}{\simeq} D^b_{coh}\Big(\overline{\mathcal{M}}_{n,d}, \overset{\substack{\text{a gerbe on } \overline{\mathcal{M}}_{n,d} \\ \downarrow}}{\mathcal{S}_L}\Big)$$

Q How are the cohomology / motives related ?

Γ - action on cohomology

- $*$ $\Gamma \circlearrowright H^*(\mathcal{M}_{n,L})$ and $H^*(\overline{\mathcal{M}}_{n,d}) \simeq H^*(\mathcal{M}_{n,L})^{\Gamma}$.

- $*$ However : $H^*(\mathcal{M}_{n,L}) \neq H^*(\mathcal{M}_{n,L})^{\Gamma}$

(Compare with $H^*(\mathcal{N}_{n,L}) = H^*(\mathcal{N}_{n,L})^{\Gamma}$ (Harder - Narasimhan))

- $*$ Isotypical decomposition :

$$\boxed{H^*(\mathcal{M}_{n,L}, \Lambda) = \bigoplus_{\kappa \in \hat{\Gamma}} H^*(\mathcal{M}_{n,L}, \Lambda)_{\kappa}}$$

Γ - action and tautological classes

$*$ We have

$$\begin{tikzcd} H^*(\mathcal{M}_{n,d}) \arrow[r, "res"] \arrow[dr] & H^*(\mathcal{M}_{n,L}) \arrow[d, phantom, "\rotatebox{90}{$\subsetneq$}"] & \circledast \\ & H^*(\mathcal{M}_{n,L})^{\Gamma} & \end{tikzcd}$$

Thm (Markman): $H^*(\mathcal{M}_{n,d})$ is generated by tautological classes, coming from Chern classes of $\mathcal{E}_{univ} \in Vec(\mathcal{M}_{n,d} \times C)$.

$\rightsquigarrow$ The strict inclusion $\circledast$ shows that $H^*(\mathcal{M}_{n,L})$ is not generated by tautological classes.

Weil pairing on Γ and cyclic covers

$*$ $\Gamma = Pic^\circ(C)[n]$ has a natural non-degenerate pairing:

$$\langle \, , \rangle : \Gamma \times \Gamma \longrightarrow \mu_n$$

$$\rightsquigarrow \quad \Gamma \xrightarrow{\sim} \hat{\Gamma}$$

$*$ $Pic^\circ(C)[n] \underset{AJ}{\simeq} H^1(C, \mathbb{Z}/_{n\mathbb{Z}}) \simeq Hom(\pi_1(C), \mathbb{Z}/_{n\mathbb{Z}})$, so $\gamma \in \Gamma$ gives rise to a cyclic cover $C_\gamma \xrightarrow{\pi} C$.

Fixed loci of Γ

- Let $\gamma \in \Gamma$. Write $\mathcal{M}_\gamma := (\mathcal{M}_{n,L})^\gamma$

- Since Γ is abelian, $\Gamma \curvearrowright \mathcal{M}_\gamma$.

- $\mathcal{M}_\gamma \xrightarrow{h_\gamma} \mathcal{A}_\gamma := \mathrm{Im}(h|_{\mathcal{M}_\gamma}) \xhookrightarrow{i_\gamma} \mathcal{A}_n$

- $d_\gamma := \mathrm{codim}_{\mathcal{A}_n}(\mathcal{A}_\gamma) = \frac{1}{2}\,\mathrm{codim}_{\mathcal{M}_{n,L}}(\mathcal{M}_\gamma)$

Hausel-Thaddeus conjecture $\Lambda = \mathbb{Q}(\zeta_n)$

Thm: (Groechenig-Wyss-Ziegler for Hodge numbers p-adic integration ; Maulik - Junliang Shen for Hodge structures perverse sheaves + vanishing cycles)

(i) $\gamma \in \Gamma \longleftrightarrow \kappa \in \hat{\Gamma}$.

$$H^*(\mathcal{M}_{n,L},\Lambda)_\kappa \underset{\text{PHS}}{\simeq} H^{*-2d_\gamma}(\mathcal{M}_\gamma,\Lambda)_\kappa(-d_\gamma)$$

(ii) $H^*(\mathcal{M}_{n,L},\Lambda) \simeq H^*_{orb}(\bar{\mathcal{M}}_{n,d},\Lambda;\alpha)$

Rmk: (ii) $= \bigoplus_{\kappa}$ (i) by definition of twisted orbifold cohomology.
+ Hausel-Thaddeus def of the gerbe α.

Some related works :

* (Loeser - Wyss) HT conjecture holds in $\widetilde{K}_0(\mathrm{CHM}(k, \Lambda))$.
motivic integration

* (Groechenig - Shiyu Shen) HT holds for KU :
Fourier-Mukai, vanishing cycles ...

$$KU(\mathcal{M}_{n,L}) \simeq KU(\bar{\mathcal{M}}_{n,d}, \delta_L)$$

(as expected from Donagi - Pantev)

* (Maulik - Shen, Kinjo - Koseki ...) HT holds for
DT-theory, Cohomological Hall algebras
BPS - cohomology when $(n, d) \neq 1$; within reach for IH^* ?

Maulik - Shen proof strategy (also for P = W !)

Goal: Construct $\hat{\beta} : (\mathbb{R}_* \Lambda)_\kappa \xrightarrow{\sim} i_{\gamma *}(\mathbb{R}_{\gamma *} \Lambda)_\kappa(-d_\gamma)[-2d_\gamma]$ in $D^b_c(\mathcal{A}_n)$

A) Work with D-twisted Higgs bundles

B) Construct a morphism in $D^b_c(\mathcal{A}^D_n)$

usual top sheaves

$$\beta^D : (R^D_* \Lambda)_\kappa \longrightarrow i_{\gamma *}(R^D_{\gamma *} \Lambda)_\kappa(-d_\gamma)[-2d_\gamma]$$

C) Show that β^D is an iso using perverse sheaves.

D) Go back to usual Higgs bundles via vanishing cycles.

A) D-twisted Higgs bundles

- Let D divisor with either $\begin{cases} \mathcal{O}_C(D) \simeq \omega_C \\ D > 0,\ \deg(D) > 2g-2 \end{cases}$

- A **D-Higgs bundle** is (E,θ) with $\theta : E \longrightarrow E \otimes \mathcal{O}_C(D)$

  Can think of (E,θ) as a meromorphic Higgs bundle with poles at the punctures D.

- Old idea, already used by Ngo in proof of fundamental lemma.

- The theory then looks the same, except when $\deg > 2g-2$:
  - $\dim(\mathcal{A}_n^D) > \frac{1}{2}\dim(\mathcal{M}_{n,d}^D)$
  - $\mathcal{M}_{n,d}^D, \mathcal{M}_{n,L}^D$ are not symplectic anymore
  - R^D has a "simpler topology":

(Bottacin, Markman: Poisson, can be odd-dim)

- $f : X \to Y$ proper morphism $/\mathbb{C}$ with X smooth

  Decomposition theorem (BBDG) $\Rightarrow$

  $${}^{p}R^i f_*(\mathbb{Q}_X[d]) \simeq \bigoplus_{\alpha} IC_{Z_\alpha}(L_\alpha) \quad \text{with } Z_\alpha \hookrightarrow Y$$

  $\{Z_\alpha\}$ supports of f.

Support theorems for twisted Higgs bundles.

Thm: Assume $\deg D > 2g-2$

(i) (Chaudouard-Laumon) $h^D : \mathcal{M}_{n,d} \to \mathcal{A}_n$ has only one support, namely $\mathcal{A}_n$. ("full support")

(ii) (de Cataldo) $h^D : \mathcal{M}_{n,L} \to \bar{\mathcal{A}}_n$ has supports $\{ \mathcal{A}^D_\gamma \}$.

Rmk: Based on Ngo's methods, but crucially apply over full $\mathcal{A}_n$.

B) Construction of β^D ($\deg D$ even $> 2g-2$)

- Recall :

$$\begin{tikzcd}
\Gamma \circlearrowright \mathcal{M}_\gamma^D = \mathcal{M}_{n,L}^{D\gamma} \arrow[r, hook, "/" marking] \arrow[d, "h_\gamma^D"'] & \mathcal{M}_{n,L}^D \circlearrowleft \Gamma \arrow[d, "h^D"] \\
\mathcal{A}_\gamma^D \arrow[r, hook, "/" marking, "i_\gamma^D"] & \bar{\mathcal{A}}_n^D = \bigoplus_{i=2}^{n} H^0(C, \mathcal{O}(iD))
\end{tikzcd}$$

Want morphism in $D_c^b(\bar{\mathcal{A}}_n^D)$

- $\beta^D : (h_*^D \Lambda)_\kappa \longrightarrow i_{\gamma *}^D (h_{\gamma *}^D \Lambda)_\kappa (-d_\gamma^D)[-2 d_\gamma^D] \in D_c^b(\bar{\mathcal{A}}_n^D)$

Endoscopic correspondence (Ngo, Yun)

- β^D is constructed using
  - ⁎ moduli of Higgs bundles on the cyclic cover C_γ
  - ⁎ an explicit correspondence coming from geometry of generic Hitchin fibers on $\mathcal{A}_\gamma^D$.

- Fits into larger pattern of endoscopy for moduli of G-Higgs bundles : the $\mathcal{A}_\gamma^D$ are related to endoscopic groups of SL_n.

C) β^D isomorphism (deg D even > 2g-2)

- β^D generically isomorphism on $\mathcal{A}^D_\gamma$: explicit computation (Ngo-Yun)
- After taking perverse cohomology objects, both sides of β^D are intermediate extensions of local systems by analysis of supports on the full Hitchin base for
  - R^D (de Cataldo)
  - R^D_π (Maulik-Shen) ← again based on Ngo's method.
- Conclude β^D iso (Ngo's "perverse continuation principle")

D) Vanishing cycles

Fix $p \in C$

* There is a ($\simeq$) quadratic form $\mu : \mathcal{A}^{D+p}_n \longrightarrow \mathbb{A}^1$ such that,

$$\mathcal{M}^D_{n,L} = \mathrm{Crit}(\mu \circ R^{D+p}) \overset{L}{\not\hookrightarrow} \mathcal{M}^{D+p}_{n,L}$$

* By applying $\varphi_\mu : D^p_c(\mathcal{A}^{D+p}_n) \longrightarrow D^b_c(\mathcal{A}^D_n)$ to β^{D+p}:

$$\hat{\beta}^D : (R^D_* \mathbb{C})_\kappa \xrightarrow{\sim} i_{\gamma *}(R^D_{\gamma *}\mathbb{C})_\kappa(-d_\gamma)[-2d_\gamma]$$

* Can go down to $D = K_C$ => the proof is complete $\square$

Motivic mirror symmetry

Thm (Hoskins - PL) In $DM(k, \Lambda)$, we have

(i) $\gamma \in \Gamma \longleftrightarrow \kappa \in \hat{\Gamma}$

$$M(\mathcal{M}_{n,L})_\kappa \simeq M(\mathcal{M}_\gamma)_\kappa(d_\gamma)[2d_\gamma]$$

(ii)

$$M(\mathcal{M}_{n,L}) \simeq M^{orb}(\bar{\mathcal{M}}_{n,d}, \alpha)$$

Cor: * Relation between (Higher) Chow groups of $\mathcal{M}_{n,L}$ and $\mathcal{M}_\gamma$.

* Categorification of Loeser - Wyss

Adapting the Maulik - Shen strategy

* **Goal:** $\hat{\beta}^{mot} : (R_* \Lambda)_\kappa \longrightarrow i_{\gamma *}(R_{\gamma *}\Lambda)_\kappa(-d_\gamma)[-2d_\gamma]$ in $DM(\mathcal{A}_n, \Lambda)$

  such that $\hat{\beta}^{mot}$ induces iso (i) (don't need $\hat{\beta}$ iso !)

* Steps A, B, D can be adapted directly to motivic sheaves:
  - A : identical
  - B : use same endoscopic correspondence and make it into a morphism in DM using (+)
  - D : relies on motivic vanishing cycles (Ayoub + ε to extend to stacks)

What about Step C?

- In an ideal motivic world, $DM(S,\Lambda)$ would admit a perverse motivic t-structure satisfying the decomposition theorem, with the same supports as in cohomology, and Step C would go through.
- At any rate, we pushforward to $\mathrm{Spec}(k)$ and get morphism:

$$(v_* \hat{\beta}^{mot})^\vee : M(\mathcal{M}_\gamma)_K(d_\gamma)[2d_\gamma] \longrightarrow M(\mathcal{M}_{n,L})_K$$

whose realisation is an isomorphism by Maulik - Shen.

Abelian motives & moduli spaces on curves

Heuristic Motives of moduli spaces of bundles on a curve C tend to be abelian, often (but not always) in $\langle M(C)\rangle^{\otimes}$.

(more generally, motives of moduli of sheaves on X often in $\langle M(X)\rangle^{\otimes}$?)

Thm (Hoskins - PL)

(i) $M(\mathcal{M}_{n,d}) \in \langle M(C)\rangle^{\otimes}$

(ii) $M(\mathcal{M}_{n,L})$ is abelian.

(iii) (H-PL-Fu) $M(\mathcal{M}_{2,L}) \notin \langle M(C)\rangle^{\otimes}$ for $C/_{\mathbb{C}}$ general $(g\geq 2)$

Idea of proof: (for (i), based on García-Prada-Heinloth-Schmitt)

- Scaling action + motivic BB decomposition reduces to study $(\mathcal{M}_{n,d})^{\mathbb{G}_m}$, which has interpretation as moduli of semistable chains of vector bundles

$$F_0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots \longrightarrow F_r$$

- Study of moduli stacks of chains, wall-crossing, Hecke stacks $\Rightarrow$ reduce to $M(\mathrm{Bun}_{n,d}) \in \langle\langle M(C) \rangle\rangle^{\otimes}$

End of the proof of motivic mirror symmetry

$$(v_* \hat{\beta}^{mot})^\vee : M(\mathcal{M}_\gamma)_K(d_\gamma)[2d_\gamma] \longrightarrow M(\mathcal{M}_{n,L})_K$$

- The RHS is abelian since $M(\mathcal{M}_{n,L})$ is.

- The LHS is abelian because, using arguments of MS, $M(\mathcal{M}_\gamma)_K$ is a direct factor of $M(\mathcal{M}_{r,d}(C_\gamma))$.

- Wildeshaus $\Rightarrow$ $(v_* \hat{\beta}^{mot})^\vee$ isomorphism $\square$

Future work and questions

- Same ideas $\Rightarrow$ $M(\mathcal{M}_{n,d}) \simeq M(\mathcal{M}_{n,d'})$

  for all d, d' with $(n,d) = (n,d') = 1$.

- $\mathrm{Char}(k) \neq 0$ should also work by lifting to char 0

  + motivic version of de Cataldo – Zhang.

- $\underline{Q} \Big|$ $M(h^{-1}(a))$ abelian for all a ?

- Going beyond $(n,d) = 1$ $\rightsquigarrow$ motivic Hall algebras.