Hecke algebras: finite, affine, double

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- Wait, this sounds familiar... didn't we *just* study Jacobi(-Heckman-Opdam) polynomials?

- Difference: the orthogonality measure is "q-deformed". Some consequences:
  - Commuting differential operators are replaced by commuting q-difference operators.
  - Affine Hecke algebras (in various guises) are replaced by Double Affine Hecke Algebras (DAHAs).
  - Links with q-calculus and with quantum groups.
  - Mysterious symmetries given by Cherednik's difference Fourier transform, extending to an action of (congruence subgroups of) $\mathrm{PSL}_2(\mathbb{Z})$.

What about Koornwinder?

- Macdonald polynomials and DAHAs for non-reduced root systems have been studied by Koornwinder (among others).
- In particular, the case $(C_1, C_1^\vee)$ recovers the Askey-Wilson orthogonal polynomials.
- The Lie-theoretic data underlying Macdonald polynomials, especially for non-reduced root systems, are quite subtle; I will only treat a simple case.

Overview of the first talk: Hecke algebras

- Lie-theoretic data
- Coxeter groups and Weil groups
- Braid groups and Hecke algebras
- Affine Hecke algebras
- Double Affine Hecke algebras

Lie-theoretic data

Root systems and root data

Definition

A root system is a f.d. $\mathbb{Q}$-vector space V and a finite spanning set $R \subset V$ such that for all $a \in R$, there is $a^\vee$ in the $\mathbb{Q}$-linear dual V^* with

- $a^\vee(R) \subset \mathbb{Z}$ *and* $(a, a^\vee) = 2$
- $s_a(R) \subset R$ *with* $s_a(v) := v - (v, a^\vee)a$.

A root datum is a quadruple $(X, R, X^\vee, R^\vee)$ with X a free abelian group of finite rank, $X^\vee$ its linear dual and $R \subset X$, $R^\vee \subset X^\vee$ finite sets such that there exists a bijection $a \mapsto a^\vee$ satisfying

- *For all $a \in R$, $(a, a^\vee) = 2$.*
- *For all $a \in R$, if we define $s_a \in \mathrm{End}(X)$ and $s_{a^\vee} \in \mathrm{End}(X^\vee)$ by*

$$s_a(x) := x - (x, a^\vee)a \text{ and } s_{a^\vee}(\lambda) := \lambda - (a, \lambda)a^\vee$$

then $s_a(R) = R$ and $s_{a^\vee}(R^\vee) = R^\vee$.

First remarks

- If $(X, R, X^\vee, R^\vee)$ is a root datum, then (V, R) (resp. $(V^\vee, R^\vee)$), where V is the span of R in $X_\mathbb{Q}$ (resp. of $R^\vee$ in $X_\mathbb{Q}^\vee$) are (dual) root systems.
- We say that $(X, R, X^\vee, R^\vee)$ is:
  - semisimple if R spans $X_\mathbb{Q}$ (equivalently $R^\vee$ spans $X_\mathbb{Q}^\vee$).
  - semisimple and adjoint if R spans X.
  - semisimple and simply connected if $R^\vee$ spans $X^\vee$.
- All the classical notions associated to root systems extend naturally to root data: reduced vs unreduced, simple/positive roots, Weyl groups...
- The definition is symmetrical: $(X^\vee, R^\vee, X, R)$ is also a root datum, the Langlands dual root datum (this is the main reason to include $X^\vee$ in the notation; some authors omit it altogether).

An abundance of lattices

Given a root datum $(X, R, X^\vee, R^\vee)$, we have

- The root lattice $Q := \mathbb{Z}R$.
- The weight lattice $P := \{x \in X_\mathbb{Q} | \forall a^\vee \in R^\vee, (x, a^\vee) \in \mathbb{Z}\}$.
- The coroot lattice $Q^\vee := \mathbb{Z}R^\vee$.
- The coweight lattice $P^\vee := \{\lambda \in X^\vee_\mathbb{Q} | \forall a \in R, (a, \lambda) \in \mathbb{Z}\}$.

We have

$$Q \subset X \subset P \text{ and } Q^\vee \subset X^\vee \subset P^\vee.$$

A semisimple root datum is simplyconnected if $X = Q$ and adjoint if $X = P$.

Conversely, if (V, R) is a root system, then it is possible to define P, Q and then construct the corresponding finitely many semisimple root data choosing a lattice $Q \subset X \subset P$.

Minuscule weights and coweights; the group Ω

Let $(X, R, X^\vee, R^\vee)$ be a root datum, and fix a decomposition $R = R^+ \subset R^-$ into positive and negative roots. This determines cones $Q^+ \subset Q, X^+ \subset X, P^+ \subset P, \ldots$.

- A weight $\lambda \in P^+$ is called minuscule if it satisfies $0 \leq (\lambda, \alpha^\vee) \leq 1$ for all $\alpha^\vee \in R^\vee$.
- (Semisimple case): a weight $\lambda \in P$ is fundamental if it is dual to a coroot $\alpha_i^\vee$: $(\lambda, \alpha_j^\vee) = \delta_{ij}$. Minuscule weights are fundamental, but not conversely in general.
- Minuscule weights form a canonical choice of representatives of P/Q, and this gives them a structure of abelian group (finite in semisimple case); in particular, for A_n, every fundamental weight is minuscule. The group $\Omega = P/Q$ plays an important role in the theory of (double) affine Hecke algebras. In type A_n, $\Omega \simeq \mathbb{Z}/n\mathbb{Z}$.

Root data, compact groups, reductive groups

- One of the main functions of root *systems* is to classify semisimple complex Lie algebras, or equivalently semisimple compact real Lie algebras. Root data allows to lift this to the group level.
- Let K be a connected compact Lie group and $T \subset K$ a maximal torus. Let $X(T) = \mathrm{Hom}(T, U(1))$ (resp. $X^\vee(T) = \mathrm{Hom}(U(1), T)$) be the character (resp. cocharacter) lattice of T. Let $R(G, T)$ (resp. $R^\vee(G, T)$) be the set of roots (resp. coroots). Then

$$(X(T), R(G, T), X^\vee(T), R^\vee(G, T))$$

is a (reduced) root datum.

Theorem

This sets up a bijection between isomorphism classes of connected compact Lie groups and reduced root data. Moreover, isomorphisms and (central) isogenies between connected compact Lie groups can be read off the root data. The same statements hold for connected split reductive groups over arbitrary fields.

Some examples in type A

- $\mathbb{G}_m^n$: $(\mathbb{Z}^n, \emptyset, \mathbb{Z}^n, \emptyset)$
- SL_2: $(\mathbb{Z}, \{\pm 2\}, \mathbb{Z}, \{\pm 1\})$.
- PGL_2: $(\mathbb{Z}, \{\pm 1\}, \mathbb{Z}, \{\pm 2\})$.
- GL_n: $(\mathbb{Z}^n, \{e_i - e_j | 1 \leq i \neq j \leq n\}, \mathbb{Z}^n, \{e_i - e_j | 1 \leq i \neq j \leq n\})$
- SL_n:
$$(\mathbb{Z}^n / \sum_i e_i, \{\bar{e}_i - \bar{e}_j | 1 \leq i \neq j \leq n\}, \mathrm{sum}^{-1}(0), \{e_i - e_j | 1 \leq i \neq j \leq n\}).$$
- PGL_n:
$$(\mathrm{sum}^{-1}(0), \{e_i - e_j | 1 \leq i \neq j \leq n\}, \mathbb{Z}^n / \sum_i e_i, \{\bar{e}_i - \bar{e}_j | 1 \leq i \neq j \leq n\}).$$

Double root data/Initial data

- The initial datum necessary to develop the theory of double affine Hecke algebras and Macdonald-Koornwinder polynomials in full generality is *two* root data whose corresponding root systems are either equal (untwisted case) or dual to each other (twisted case), and where that identification at that level of root systems is extended in a natural way to the root datum level.
- In the Askey-Bateman volume, this is encoded by a notion of initial datum; similar concepts are used by Macdonald, Haiman, Ion-Sahi in other papers.
- Much too general for an introductory talk! We will only consider the untwisted, adjoint simply connected case attached to an irreducible root system R of some Dynkin type A-G. This corresponds to restricting to a large and interesting class of affine root systems.
- This excludes the case of Koornwinder polynomials ;-(.

Braid groups and Hecke algebras

Coxeter groups

- A Coxeter group (W, S) is a group generated by $S = \{s_i | i \in I\}$ with the relations $(s_i s_j)^{m(i,j)} = 1$ for some matrix $m(i, j)$ with $m(i, i) = 1$ and $m(i, j) \geq 2$ for $i \neq j$.
- An alternative presentation is given by the quadratic relations $s_i^2 = 1$ and, for $i \neq j$, the braid relations

$$s_i s_j s_i \ldots = s_j s_i s_j \ldots$$

  with $m(i, j)$ factors on each side.
- Any Coxeter group admits a faithful real representation in which the s_i's act by reflections. Moreover, any *finite* reflection group is a Coxeter group for some choice of S.
- Weyl groups of root data are Coxeter groups. Conversely, irreducible finite Coxeter groups are either Weyl groups, dihedral groups, or the symmetry groups H_3 and H_4 of the icosahedron and the 120-cell.
- Besides finite Weyl groups, we are interested in affine Weyl groups.

Braid groups

Let $W = (W, S)$ be a Coxeter group.

- Any two reduced expressions of $w \in W$, i.e. of minimal length $l(w)$, in terms of the s_i's are connected by the braid relations (Matsumoto's theorem).
- The braid group B_W is generated by S together with the braid relations. By construction there is a surjection $B_W \twoheadrightarrow W$.
- By Matsumoto, one can define a lift $T_w \in B_w$ of any $w \in W$. These generate B_W with the relations $T_w T_{w'} = T_{ww'}$ when $l(ww') = l(w) + l(w')$.
- When $W = S_n$ Weyl group of type A_{n-1}, B_W is the classical braid group, i.e. the fundamental group of the configuration space of n (unordered, distinct) points in $\mathbb{C}$.
- More generally, for W a finite Weyl group with $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra of the corresponding semisimple complex Lie algebra, we have $B_W \simeq \pi_1(\mathfrak{h}^{\mathrm{reg}}/W)$ with $\mathrm{h}^{\mathrm{reg}}$ the regular elements.

Hecke algebras

Let (W, S) be a Coxeter group. Let R be a base ring (in practice often $R = \mathbb{C}$). Fix formal variables q_s for $s \in S$, with $q_s = q_{ws}$ for all $w \in W$. We can also take one $q_s = q$ (automatic when there is one root length).

- The (Iwahori-)Hecke algebra $H(W, S)$ over R is the quotient of the group algebra $R[q_s, q_s^{-1}][B_W]$ by Hecke relations (or q-deformed quadratic relations):

  $$(T_s - q_s)(T_s + q_s^{-1}) = 0.$$

  or alternatively

  $$T_s - T_s^{-1} = q_s - q_s^{-1}$$

- We can also specialize q_s to fixed invertible elements in R. For $q_s = 1$, we recover the group algebra $R[W]$ of the Coxeter group.
- This is a flat deformation of $R[q_s, q_s^{-1}][W]$; for $R = \mathbb{C}$ and fixed q this just means that the elements T_w for $w \in W$ form a basis of $H(W, S)$ as a vector space.

Finite Hecke algebras

- When W is a (finite) Weyl group, $H(W)$ is called a (finite) Hecke algebra. It is then a finite dimensional deformation of the group algebra $R[W]$.
- $H(W)$ with fixed $q \in \mathbb{C}^\times$ has a quite simple representation theory; for all but finitely many q's, it is the same as the one of W.
- Finite Hecke algebras occur naturally in the study of reductive groups over finite fields and their representations: if G is a split reductive group over $\mathbb{F}_q$ and B a Borel subgroup, then after specialising to $q_s = q$, $H(W)$ is isomorphic to the algebra of B-biinvariant functions $G \to \mathbb{C}$ equipped with its natural convolution product (follows from Bruhat decomposition).
- Finite Hecke algebras also appear in quantum groups, knot theory, combinatorics,...

Affine Hecke algebras

Affine root systems and affine Weyl groups

- As warned, we only consider a somewhat special class of affine root systems. Let $(P, R, P^\vee, R^\vee)$ be an irreducible semisimple adjoint root datum.
- Let $V = P_\mathbb{R}$. We have affine hyperplanes:

  $$H_{\alpha,n} = \{x \in V | (x, \alpha^\vee) = n\}.$$

  The reflections along those hyperplanes form the affine Weyl group W^a; we have $W \subset W^a$, and in fact $W^a \simeq Q \rtimes W$.
- Fix a set of simple roots $S \subset R$, so that (W, S) is a Coxeter group. Let $\theta \in R_+$ be the corresponding highest root, and $s_0 = s_{H_{\theta,1}}$. Then $(W^a, s_0 \cup S)$ is a Coxeter group.
- Let $W^{ae} := P^\vee \rtimes W$ be the extended affine Weyl group. Then $W^{ae} \simeq W^a \rtimes \Omega$ with $\Omega = P^\vee / Q^\vee$. W^{ae} is not Coxeter, but one can extend the length function of W^a to it so that $\Omega = l^{-1}(0)$.

Extended affine braid group: definition

- The affine braid group $B_W^a := B_{W^a}$ also has a variant taking into account the larger, non-Coxeter group W^{ae}.
- The extended affine braid group B_W^{ae} is generated by $\{T_w | w \in W^{ae}\}$ with the relations $T_w T_{w'} = T_{ww'}$ when $l(ww') = l(w) + l(w')$ (with the extended length function).
- The elements $\{T_w | w \in \Omega\}$ form a subgroup isomorphic to Ω, and we have

$$B_W^{ae} = B_W^a \rtimes \Omega.$$

Extended affine braid group: Bernstein presentation

Choose a decomposition $R = R^+ \cup R^-$, which gives rise to a set of simple roots $S = \{\alpha_1, \ldots, \alpha_n\}$.

- For $\lambda \in P$, one can introduce $Y^\lambda \in B_W^{ae}$ by:
  - $Y^\lambda := T_\lambda$ for $\lambda \in P^+$
  - $Y^\lambda := T_\mu (T_\nu)^{-1}$ for any equality $\lambda = \mu - \nu$ with $\mu, \nu \in P^+$.

  and check that this is well-defined.
- This produces a subgroup $Y^P := \{Y^\lambda | \lambda \in P\} \subset B_W^{ae}$ isomorphic to P.
- A key point in the whole story is the Bernstein presentation of B_W^{ae}: the group B_W^{ae} is generated by Y^P and $T_1, \ldots, T_n$ with relations:
  - If $(\lambda, \alpha_i^\vee) = 0$, then $Y^\lambda T_i = T_i Y^\lambda$.
  - If $(\lambda, \alpha_i^\vee) = 1$, then $Y^\lambda = T_i Y^{s_i \lambda} T_i$.

Affine Hecke algebras: definition

- We have the Hecke algebra $H(W^a)$ of the affine Weyl group W^a with the above choice of Coxeter system. It is already an interesting algebra, however it is not what is usually called the affine Hecke algebra!
- Instead, we replace the affine braid group B_W^a by the extended affine braid group B_W^{ae}. The affine Hecke algebra $H(W^{ae})$ is the quotient of $R[q_s, q_s^{-1}][B_W^a]$ by q-deformed quadratic relations:

$$(T_s - q_s)(T_s + q_s^{-1}) = 0.$$

or alternatively

$$T_s - T_s^{-1} = q_s - q_s^{-1}$$

- As with braid groups, we have $H(W^{ae}) \simeq H(W^a) \rtimes \Omega$.
- The algebras $H(W^{ae})$ are the ones that appears in the representation theory of p-adic groups, and in the theory of double affine Hecke algebras.

Affine Hecke algebras: Bernstein presentation

- The $R[q_s^{\pm 1}]$-subalgebra of $H(W^{ae})$ generated by $T_1, \ldots, T_n$ is isomorphic to $H(W)$.
- The $R[q_s^{\pm 1}]$-submodule generated by Y^P is isomorphic to the group algebra of the weight lattice $R[q_s^{\pm 1}][Y^P]$.
- Multiplication then gives rise to an isomorphism of $R[q_s^{\pm 1}]$-modules

$$H(W) \otimes_{R[q_s^{\pm 1}]} R[q_s^{\pm 1}][Y^P] \simeq H(W^{ae}).$$

- Moreover, the commutation relations between those two subalgebras are given by, for $1 \leq i \leq n$ and $\lambda \in P$:

$$T_i Y^\lambda - Y^{s_i \lambda} T_i = (q_i - q_i^{-1}) \frac{Y^{s_i \lambda} - Y^\lambda}{Y^{-\alpha_i} - 1}.$$

- This implies that the center of $H(W^{ae})$ is $R[q_s^{\pm 1}][Y^P]^W$.

Affine Hecke algebras: polynomial representation

- The representation theory of affine Hecke algebras is complicated and has been extensively studied, but we only need very special representations.
- One natural construction is by inducing a finite-dimensional representation E of $H(W)$ along the inclusion $H(W) \subset H(W^{ae})$. By the previous slide, the resulting representation $\mathrm{Ind}_{H(W)}^{H(W^{ae}} E$ has representation space $R[q_s^{\pm 1}][Y^P] \otimes_{R[q_s^{\pm 1}]} E$. For $E = R[q_s^{\pm 1}]$ the "trivial" representation $T_i \to q_i$, we get the polynomial representation of $H(W^{ae})$ on $R[q_s^{\pm 1}][Y^P]$.
- The commutation relations imply that in the polynomial representation, the subalgebra $R[q_s^{\pm 1}][Y^P]$ just acts by multiplication, while T_i for $1 \leq i \leq n$ acts by

$$T_i \mapsto q_i s_i + (q_i - q_i^{-1}) \frac{s_1 - 1}{Y^{-\alpha_i} - 1}.$$

Double affine Hecke algebras

Double affine Weyl groups

- The pairing $P \times P^\vee \to \mathbb{Q}$ does not necessarily take integer values, but we can fix $e \geq 1$ such that $P \times P^\vee \to \frac{1}{e}\mathbb{Z}$.
- Write $\tilde{P} := P \oplus \frac{1}{e}\mathbb{Z}\delta$ and $\tilde{P}^\vee := P \oplus \frac{1}{e}\mathbb{Z}\delta'$ so that $\tilde{P}$ and $\tilde{P}^\vee$ together with the simple roots $\alpha_0, \dots, \alpha_n$ and $\alpha_0^\vee, \dots, \alpha_n^\vee$ form affine root systems of the type considered above.
- The extended affine Weyl groups $W^{ae} := P \rtimes W$ (resp. $W^{ae}_\vee := P^\vee \rtimes W$) act on $\tilde{P}^\vee$ (resp. $\tilde{P}$) respectively via

$$\lambda \cdot \mu := \mu - (\mu, \lambda)\delta' \text{ (resp....)}$$

- We can now form two extended double affine Weyl groups W^{dae} and $W^{dae}_\vee$ defined as $W^{dae} := P^\vee \rtimes W^{ae}$ and $W^{dae}_\vee := P \rtimes W^{ae}_\vee$.
- There is a canonical isomorphism $W^{dae} \simeq W^{dae}_\vee$ which is the identity on $P, P^\vee, W$ and sends δ to $-\delta'$.

Double affine braid groups

- Using the Coxeter group W^a, its extension W^{ae} and its action on $\tilde{P}^\vee$, we can take inspiration from the Bernstein presentation of $B_{W^{ae}}$ to define a (left) double affine braid group $B(W^{ae}, \tilde{P}^\vee)$ as the group generated by $B(W^{ae})$ and $X^{\tilde{P}^\vee} \simeq \tilde{P}^\vee$ with the additional relations
  - If $(\alpha_i, \lambda) = 0$, then $X^\lambda T_i = T_i X^\lambda$.
  - If $(\alpha_i, \lambda) = 1$, then $X^\lambda = T_i X^{s_i \lambda} T_i$.
- Similarly we have a (right) double affine braid group $B(\tilde{P}^\vee, W^{ae}_\vee)$ as the group generated by $B(W^{ae}_\vee)$ and $Y^P \simeq P$ together with the relations
  - If $(\lambda, \alpha_i^\vee) = 0$, then $Y^\lambda T_i = T_i Y^\lambda$.
  - If $(\lambda, \alpha_i^\vee) = 1$, then $Y^\lambda = T_i^{-1} Y^{s_i \lambda} T_i^{-1}$.

Cherednik duality theorem

Theorem

The isomorphism $W^{dae} \simeq W^{dae}_{\vee}$ lifts to an isomorphism of double affine braid groups

$$B(W^{ae}, \tilde{P}^{\vee}) \simeq B(\tilde{P}^{\vee}, W^{ae}_{\vee})$$

which is the identity on P, $P^{\vee}$ and the braid group $B(W)$ (which are all subgroups in a natural way) and maps X^{δ} to $Y^{\delta'}$.

Double affine Hecke algebra

- We can now define the (left) DAHA $H(W^{ae}, \tilde{P}^\vee)$ taking the Bernstein presentation and replacing P by $\tilde{P}^\vee$. I.e. it is the algebra over $R[q_s^{\pm 1}]$ generated by $X^{\tilde{P}^\vee}$, $T_0, \ldots, T_n$ and Ω satisfying the relations of the left double affine braid group together with the quadratic relations

$$(T_i - q_i)(T_i + q_i^{-1}) = 0$$

- Similarly, we can define the right DAHA $H(\tilde{P}, W^{ae}_\vee)$ dually using the right double affine braid group. (There is a little reindexing of the q_i's which I won't go into right now).

Theorem

There is an isomorphism $H(W^{ae}, \tilde{P}^\vee) \simeq H(\tilde{P}, W^{ae}_\vee)$ which is the identity on all generators X, Y, T_i, Ω...

This common algebra is Cherednik's DAHA, and this dual incarnation is one of the key properties to study Macdonald polynomials, which we will do next time.