INFINITY CATEGORIES SEMINAR

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0. Introduction

Our goal is to define limits and colimts in infinity categories. We will motivate these definitions with the ordinary categorical definitions.

Definition 0.1. The ∞-category **Top** of topological spaces is the following simplicial set: an n-simplex is:

(1) A tuple $(X_0, \dots, X_n)$ of $n+1$ topological spaces.
(2) A family of morphisms
$$\big(h_{i,j} : X_i \times \square^{j-i-1}_{\text{top}} \to X_j\big)_{0 \le i < j \le n}$$
where $\square^m_{\text{top}} := \{(t_1, \dots, t_m) \in \mathbb{R}^m : 0 \le t_i \le 1\}$.
(3) The morphisms $h_{i,j}$ are required to satisfy the compatibility condition: for every $0 \le i < j < k \le n$, we should have
$$h_{i,k}\big(x, (s_1, \dots, s_{j-i-1}, 1, t_1, \dots, t_{k-j-1})\big) = h_{j,k}\big(h_{i,j}(x, (s_1, \dots, s_{j-i-1})), (t_1, \dots, t_{k-j-1})\big)$$

The face morphisms are
$$d_k : (X_0, \dots, X_n, h_{i,j}) \mapsto (X_0, \dots, X_{k-1}, X_{k+1}, \dots, X_n, h'_{i,j})$$
where
$$h'_{i,j}(x, y) := \begin{cases} h_{i,j}(x,t) & \text{if } i < j < k \\ h_{i,j+1}\big(x, (t_1, \dots, t_{k-i-1}, 0, t_{k-i}, \dots, t_{j-i-1})\big) & \text{if } i < k \le j \\ h_{i+1,j+1}(x,t) & \text{if } k \le i < j \end{cases}$$

The degeneracy morphisms are
$$s_k : (X_0, \dots, X_n, h_{i,j}) \mapsto (X_0, \dots, X_k, X_k, \dots, X_n, h'_{i,j})$$
where
$$h'_{i,j}(x, t) := \begin{cases} h_{i,j}(x,t) & \text{if } i < j \le k \\ h_{i,j-1}\big(x, (t_1, \dots, t_{k-i-1}, t_{k-i+1}, \dots, t_{j-i-1})\big) & \text{if } i \le k < j \\ h_{i-1,j-1}(x,t) & \text{if } k < i < j \end{cases}$$
where we interpret $h_{i,i}$ as id_{X_i}.

Note that every sequence of continuous maps $X_0 \xrightarrow{f_1} X_1 \to \cdots \xrightarrow{f_n} X_n$ defines an n-simplex: choose $h_{i,j}$ to be the composition $X_i \times \square^{j-i-1}_{\text{top}} \to X_i \xrightarrow{f_{i+1}} X_{i+1} \to \cdots \xrightarrow{f_j} X_j$.

Definition 0.2. A morphism $p : K \to \mathcal{C}$ of simplicial sets is a *weak equivalence* if the geometric realization is a homotopy equivalence.

Definition 0.3. Let $\mathcal{C}$ be a category and let $p : A \to B$ and $q : X \to Y$ be morphisms in $\mathcal{C}$. We say that p has the *left lifting property* with respect to q, and q has the right lifting property with respect to p, if given any diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle p}\downarrow & \nearrow & \downarrow{\scriptstyle q} \\ B & \longrightarrow & Y \end{array}$$

there exists a dotted arrow as indicated, rendering the diagram commutative.

Definition 0.4. A morphism $p : K \to \mathcal{C}$ of simplicial sets is a *Kan fibration* if it has the right lifting property with respect to every horn inclusion $\Lambda_i^n \subseteq \Delta^n$.

Definition 0.5. If $\mathcal{C}$ is an ordinary category, we define $\mathcal{C}^{\text{op}}$ by: the objects of $\mathcal{C}^{\text{op}}$ are the same as $\mathcal{C}$, and for $x, y \in \text{obj}(\mathcal{C})$, we define $\hom_{\mathcal{C}^{\text{op}}}(x, y) := \hom_{\mathcal{C}}(y, x)$.

Definition 0.6. The *opposite of a simplicial set* S as follows: we set $S_n^{\text{op}} = S_n$, but the face and degeneracy maps on S^{op} are given by the formulas

$$(d_i : S_n^{\text{op}} \to S_{n-1}^{\text{op}}) = (d_{n-i} : S_n \to S_{n-1})$$
$$(s_i : S_n^{\text{op}} \to S_{n+1}^{\text{op}}) = (s_{n-i} : S_n \to S_{n+1}).$$

Proposition 0.7. A simplicial set S is an ∞-category if and only if its opposite category S^{op} is a ∞-category.

Proof. Omitted. See Lurie, HTT, p. 26. □

1. Join and Splice ∞-categories

Definition 1.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be ordinary categories. We will define a new category $\mathcal{C} * \mathcal{C}'$, called the *join* of $\mathcal{C}$ and $\mathcal{C}'$. An object of $\mathcal{C} * \mathcal{C}'$ is either an object of $\mathcal{C}$ or an object of $\mathcal{C}'$. The morphism sets are given as follows:

$$\hom_{\mathcal{C}*\mathcal{C}'}(x, y) := \begin{cases} \hom_{\mathcal{C}}(x, y) & \text{if } x, y \in \mathcal{C} \\ \hom_{\mathcal{C}'}(x, y) & \text{if } x, y \in \mathcal{C}' \\ \emptyset & \text{if } x \in \mathcal{C}', y \in \mathcal{C} \\ * & \text{if } x \in \mathcal{C}, y \in \mathcal{C}'. \end{cases}$$

Definition 1.2. If S and S' are simplicial sets, then the simplicial set $S * S'$, called the *join* is defined by

$$(S * S')_n := S_n \cup S'_n \cup \bigcup_{i+j=n-1} S_i \times S'_j.$$

Proposition 1.3. The nerve is compatible with the join constructions in that there is a natural isomorphism $N(A) * N(B) \to N(A * B)$, $A, B \in \mathbf{Cat}$.

Proof. Omitted. □

Proposition 1.4. If S and S' are ∞-categories, then $S * S'$ is an ∞-category.

Proof. Omitted. □

Notation 1.5. Let K be a simplicial set. The *left cone*, or *cone* $K^{\triangleleft}$ is defined to be the join $\Delta^0 * K$. Dually, the *right cone*, or *co-cone* $K^{\triangleright}$ is defined to be the join $K * \Delta^0$. Either cone contains a distinguished vertex (belonging to Δ^0), which we will refer to as the *cone point*.

Proposition 1.6. (1) For the standard simplices we find $\Delta^i * \Delta^j \cong \Delta^{i+j+1}$ for $i, j \geq 0$, and these isomorphisms are compatible with the obvious inclusions of Δ^i and Δ^j.

(2) $(\partial\Delta^{n-1})^{\triangleleft} \cong \Lambda^n_0$.
(3) $(\partial\Delta^{n-1})^{\triangleright} \cong \Lambda^n_n$.

Definition 1.7. If $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ are functors, then their *comma category* is the category $(F \downarrow G)$ whose

- **objects** are triples (c, α, d) where $c \in \mathrm{obj}(\mathcal{C})$, $d \in \mathrm{obj}(\mathcal{D})$, and $\alpha : F(c) \to G(d)$ is a morphism in $\mathcal{E}$, and whose
- **morphisms** from (c, α, d) to (c', α', d') are pairs (β, γ), where $\beta : c \to c'$ and $\gamma : d \to d'$ are morphisms in $\mathcal{C}$ and $\mathcal{D}$, respectively, such that the following diagram commutes:

$$\begin{CD} F(c) @>{F(\beta)}>> F(c') \\ @V{\alpha}VV @VV{\alpha'}V \\ G(d) @>>{G(\gamma)}> G(d') \end{CD}$$

Definition 1.8. A special case of a comma category is the *over category* $\mathcal{C}_{/x}$ of a category $\mathcal{C}$ over an object $x \in \mathrm{obj}(\mathcal{C})$, where $\mathcal{C}_{/x} := (F \downarrow G)$, where $F : \mathcal{C} \to \mathcal{C}$ is the identity functor and $G : \mathbf{1} \to \mathcal{C}$ is defined by $* \mapsto x$ (where $\mathbf{1}$ is category with one object and one morphism). To explicitly describe this, we have that

- **objects** are morphisms $\alpha \in \mathcal{C}$ such that $\mathrm{cod}(\alpha) = x$, that is, morphisms in $\mathcal{C}$ of the form $\alpha : y \to x$, and whose
- **morphisms** are $\beta : y \to y'$ in $\mathcal{C}$ from $\alpha : y \to x$ to $\alpha' : y' \to x$ such that the following diagram commutes:

$$\begin{array}{ccc} y & \xrightarrow{\beta} & y' \\ & {\scriptstyle\alpha}\searrow & \downarrow{\scriptstyle\alpha'} \\ & & x \end{array}$$

We often write for simplicity

$$\mathcal{C}_{/x} = \left\{ \begin{array}{ccc} y & \xrightarrow{\beta} & y \\ & {\scriptstyle\alpha}\searrow & \downarrow{\scriptstyle\alpha'} \\ & & x \end{array} \right\}$$

This is sometimes called the *slice category*. Note that Groth reserves that term for the category of cones over a particular functor.

Definition 1.9. Another special case of the comma category is the *under category* $\mathcal{C}_{x/}$ of a category $\mathcal{C}$ under an object $x \in \mathrm{obj}(\mathcal{C})$, where $\mathcal{C}_{x/} := (F \downarrow G)$, where $F : \mathbf{1} \to \mathcal{C}$ is defined by $* \mapsto x$, and $G : \mathcal{C} \to \mathcal{C}$ is the identity functor. To explicitly describe this, we have that

- **objects** are morphisms $\alpha \in \mathcal{C}$ such that $\mathrm{dom}(\alpha) = x$, that is, morphisms in $\mathcal{C}$ of the form $\alpha : x \to y$, and whose
- **morphisms** are $\beta : y \to y'$ in $\mathcal{C}$ from $\alpha : x \to y$ to $\alpha' : x \to y'$ such that the following diagram commutes:

$$\begin{array}{ccc} x & & \\ {\scriptstyle\alpha}\downarrow & \searrow{\scriptstyle\alpha'} & \\ y & \xrightarrow[\beta]{} & y' \end{array}$$

We often write for simplicity

$$\mathcal{C}_{x/} := \left\{ \begin{array}{ccc} x & & \\ {\scriptstyle\alpha}\downarrow & \overset{\alpha'}{\searrow} & \\ y & \xrightarrow[\beta]{} & y' \end{array} \right\}$$

This is often called the *coslice category.*

Definition 1.10. If $F : J \to \mathcal{C}$ is a diagram, that is, a functor, then we define a *cone* of F to be a natural transformation $\Delta(b) \to F$, where $b \in \mathcal{C}$ and $\Delta : \mathcal{C} \to \mathcal{C}^J$ the functor with $b \mapsto \Delta(b)$, with $\Delta(b) : J \to \mathcal{C}$, the constant functor at b, that is $x \mapsto b$ for all $x \in J$. In other words, a cone of F is a family of morphisms $\big(\tau_x : \Delta(b)(x) = b \to F(x)\big)$ such that the following diagram commutes for all $f : x \to y$ in J:

$$\begin{array}{ccc} b & & \\ {\scriptstyle\tau_x}\downarrow & \overset{\tau_y}{\searrow} & \\ F(x) & \xrightarrow[F(f)]{} & F(y) \end{array}$$

Definition 1.11. If $F : J \to C$ is a diagram, then we define a *co-cone* of F to be a natural transformation $F \to \Delta(b)$ for some $b \in \mathcal{C}$. Equivalently, a co-cone is a family of morphisms $\big(\sigma_x : F(x) \to \Delta(b)(x) = b\big)$ such that the following diagram commutes for all $f : x \to y$ in J:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ & \underset{\sigma_x}{\searrow} & \downarrow{\scriptstyle\sigma_y} \\ & & b \end{array}$$

Definition 1.12. Let $p : J \to \mathcal{C}$ be a functor. We define the category of cones over p to be the comma category $(\Delta \downarrow p)$. This is a slight abuse of notation: we have $\Delta : \mathcal{C} \to \mathcal{C}^J$, but $p : J \to \mathcal{C}$, where for the comma category to make sense, both functors must have the same codomain. In this case, we consider p as the functor $\mathbf{1} \to \mathcal{C}^J$ defined by $* \mapsto p$.

- The **objects** of this category are cones over p as defined above: natural transformations $\tau : \Delta(b) \to p$ for some $b \in \mathcal{C}$.
- The **morphisms** of this category are morphisms $\alpha : b \to c$ in $\mathcal{C}$ between $\tau : \Delta(b) \to p$ and $\tau' : \Delta(c) \to p$, that is, such that the following diagrams commute for all $x \in J$:

$$\begin{array}{ccc} b & \xrightarrow{\alpha} & c \\ & \underset{\tau_x}{\searrow} & \downarrow{\scriptstyle\tau'_x} \\ & & p(x) \end{array}$$

We denote this category by $\mathcal{C}_{/p}$ and also call is the over category over p. Note that we recover the previous definition of over category by choosing p appropriately.

Definition 1.13. Let $p : J \to \mathcal{C}$ be a functor. We define the category of co-cones under p to be the comma category $(p \downarrow \Delta)$, where we have the same abuse of notation as before.

- The **objects** of this category are co-cones under p as defined above: natural transformations $\sigma : p \to \Delta(b)$ for some $b \in \mathrm{obj}(\mathcal{C})$.

- The **morphisms** of this category are morphisms $\alpha : b \to c$ in $\mathcal{C}$ between $\sigma : p \to \Delta(b)$ and $\sigma' : p \to \Delta(c)$, that is, such that the following diagrams commute for all $x \in J$:

$$\begin{array}{ccc} p(x) & & \\ {\scriptstyle \sigma_x}\downarrow & \searrow^{\sigma'_x} & \\ b & \xrightarrow[\alpha]{} & c \end{array}$$

We denote this category by $\mathcal{C}_{p/}$ and also call it the under category of p. Again, we can recover the definition of under category by a particular choice of p.

Proposition 1.14. Let $p : L \to \mathcal{C}$ be a map of simplicial sets with $\mathcal{C}$ an ∞-category. There is an ∞-category $\mathcal{C}_{/p}$ characterized by the following universal property: For every simplicial set K, there is a natural bijection

$$\hom_{\mathbf{sSet}}(K, \mathcal{C}_{/p}) \cong \hom_{\mathbf{sSet}_{L/}}(L \to K * L, L \to \mathcal{C}) \cong \hom_p(K * L, \mathcal{C})$$

where

$$\hom_p(K * L, \mathcal{C}) \cong \left\{ \begin{array}{ccc} L & & \\ \downarrow & \searrow^{p} & \\ K * L & \longrightarrow & \mathcal{C} \end{array} \right\}.$$

The ∞-category $\mathcal{C}_{/p}$ is the *∞-category of cones on p.*

The Yoneda lemma gives us a description of the n-simplices of $\mathcal{C}_{/p}$ as

$$(\mathcal{C}_{/p})_n \cong \hom_p(\Delta^n * L, \mathcal{C}).$$

Proposition 1.15. Let $p : L \to \mathcal{C}$ be a map of simplial sets with $\mathcal{C}$ an ∞-category. There is an ∞-category $\mathcal{C}_{p/}$ characterized by the following universal property: For every simplicial set K, there is a natural bijection

$$\hom_{\mathbf{sSet}}(K, \mathcal{C}_{p/}) \cong \hom_{\mathbf{sSet}_{L/}}(L \to L * K, L \to \mathcal{C}) \cong \hom_p(L * K, \mathcal{C})$$

where

$$\hom_p(L * K, \mathcal{C}) \cong \left\{ \begin{array}{ccc} L & & \\ \downarrow & \searrow^{p} & \\ L * K & \longrightarrow & \mathcal{C} \end{array} \right\}.$$

The ∞-category $\mathcal{C}_{p/}$ is called the *∞-category of co-cones on p.*

Remark 1.16. We often consider the special case: let $\mathcal{C}$ be an ∞-category and let $x \in \mathcal{C}$ be an object, classified by the map $\kappa_x : \Delta^0 \to \mathcal{C}$. Then the ∞-category $\mathcal{C}_{/\kappa_x}$ is called the *∞-category of objects over x*, and is simply denoted $\mathcal{C}_{/x}$. Dually, the ∞-category $\mathcal{C}_{\kappa_x/}$ is called the *∞-category of objects under x*, and is denoted by $\mathcal{C}_{x/}$.

Proposition 1.17. If $p : A \to B$ is a functor, then there is a natural isomorphism of simplicial sets

$$N(B/p) \cong N(B)_{/N(p)}.$$

2. Initial and Terminal Objects

Definition 2.1. A morphism $p : X \to S$ of simplicial sets which has the right lifting property with respect to every inclusion $\partial\Delta^n \subseteq \Delta^n$ is called a *trivial fibration* or *acyclic fibration.*

Proposition 2.2. A morphism of simplicia sets is a Kan fibration and a weak equivalence if and only if it is a trivial fibration.

Definition 2.3. Let $\mathcal{C}$ be a simplicial set. A vertex x of $\mathcal{C}$ is *final* if the projection $\mathcal{C}_{/x} \to \mathcal{C}$ is a trivial fibration of simplicial sets.

The projection is defined on the n-cells as follows: we view $(\mathcal{C}_{/x})_n \to \mathcal{C}_n$ as

$$\hom_p(\Delta^n * \Delta^0, \mathcal{C}) \cong \hom_{\mathbf{sSet}}(\Delta^n, \mathcal{C}_{/x}) \to \hom_{\mathbf{sSet}}(\Delta^n, \mathcal{C}).$$

by the adjunction between slice and join and by the Yoneda lemma, where $p : \Delta^0 \to \mathcal{C}$ defined by $p(0) = x$ (again, we get this from the correspondence of the Yoneda lemma: $x \in \mathcal{C}_n \leftrightarrow p : \Delta^0 \to \mathcal{C}$ with $p(0) = x$). Notice that

$$\hom_p(\Delta^n * \Delta^0, \mathcal{C}) = \{\sigma : \Delta^{n+1} \to \mathcal{C} : \sigma(n+1) = x\}.$$

Hence, we define our map $(\mathcal{C}_{/x})_n \to \mathcal{C}_n$ by $\big((\sigma : \Delta^{n+1} \to \mathcal{C}) : \sigma(n+1) = x\big) \mapsto \sigma|_{\Delta^{\{0,1,\ldots,n\}}}$.

Definition 2.4. Given a simplicial set S and two vertices $x, y \in S$, we define a new simplicial set $\hom_S^R(x, y)$, the space of *right morphisms* from x to y, by letting $\hom_{\mathbf{sSet}}\big(\Delta^n, \hom_S^R(x, y)\big)$ denote the set of all $z : \Delta^{n+1} \to S$ such that $z|_{\Delta^{\{n+1\}}} = y$ and $z|_{\Delta^{\{0,\ldots,n\}}}$ is the constant simplex at the vertex x.

This simplicial set can also be interpreted as the pullback of the following diagram

$$\begin{array}{ccc} & & S_{/y} \\ & & \downarrow \\ \Delta^0 & \longrightarrow & \mathcal{S} \end{array}$$

where $\Delta^0 \to \mathcal{S}$ corresponds to the vertex x.

Proposition 2.5. The following are equivalent for an object x of an ∞-category $\mathcal{C}$:
(1) The object x is final.
(2) The mapping spaces $\mathrm{map}_{\mathcal{C}}^R(x', x)$ are contractible for all $x' \in \mathcal{C}$.
(3) Every simplicial sphere $\alpha : \partial\Delta^n \to \mathcal{C}$ such that $\alpha(n) = x$ can be filled to an entire n-simplex $\Delta^n \to \mathcal{C}$.

Definition 2.6. Let $\mathcal{C}$ be a simplicial set. A vertex x of $\mathcal{C}$ is *initial* if the projection $\mathcal{C}_{x/} \to \mathcal{C}$ is a trivial fibration.

This time the projection, defined on n-cells $(\mathcal{C}_{x/})_n \to \mathcal{C}_n$, is defined by $\sigma \mapsto \sigma|_{\Delta^{\{1,2,\ldots,n+1\}}}$, using the same reasoning as before.

Definition 2.7. Given a simplicial set S and two vertices $x, y \in S$, we define a new simplicial set $\hom_S^L(x, y)$, the space of *left morphisms* from x to y, by letting $\hom_{\mathbf{sSet}}\big(\Delta^n, \hom_S^L(x, y)\big)$ denote the set of all $z : \Delta^{n+1} \to S$ such that $z|_{\Delta^{\{0\}}} = x$ and $z|_{\Delta^{\{1,\ldots,n+1\}}}$ is the constant simplex at the vertex y.

This simplicial set can also be interpreted as the pullback of the following diagram

$$\begin{array}{ccc} & & S_{y/} \\ & & \downarrow \\ \Delta^0 & \longrightarrow & S \end{array}$$

where $\Delta^0 \to S$ corresponds to x.

Proposition 2.8. The following are equivalent for an object x of an ∞-category $\mathcal{C}$:
(1) The object x is initial.
(2) The mapping spaces $\mathrm{map}_{\mathcal{C}}^L(x, x')$ are contractible for all $x' \in \mathcal{C}$.

(3) Every simplicial sphere $\alpha : \partial\Delta^n \to \mathcal{C}$ such that $\alpha(0) = x$ can be filled to an entire n-simplex.

Proposition 2.9. An ∞-category of a poset $(P, \leq)$ has a final (initial) object if and only if the poset has a maximal (minimal) object.

Proof. Suppose x is a final object in the ∞-category $N(P, \leq)$, then x is an element of P and $N(P)_{/x} \to N(P)$ is a trivial fibration, that is, for every n, we have the right lifting property with respect to the inclusion $\partial\Delta^n \to \Delta^n$:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & N(P)_{/x} \\ \downarrow & \nearrow & \downarrow \text{can} \\ \Delta^n & \longrightarrow & N(P) \end{array}$$

Notice that, by Proposition 1.17, $N(P)_{/x} \cong N(P_{/x})$. Let $y \in P$ be any element. By the Yoneda lemma, this corresponds to a map between simplicial sets $\Delta^0 \to N(P)$. Consider the above commutative square with $n = 0$. We immediatly have that $\partial\Delta^0 = \emptyset$, so the lifting property reduces to:

$$\begin{array}{ccc} & & N(P_{/x}) \\ & \nearrow & \downarrow \\ \Delta^0 & \longrightarrow & N(P) \end{array}$$

But the lifting property here tells us that $y \to x$ is an object in $P_{/x}$, that is $y \leq x$. Since $y \in P$ was arbitrary, conclude that x is a maximal element of p.

Conversely, suppose $x \in (P, \leq)$ is a maximal element. Then, we immediately have that $P_{/x} \cong P$ which means $N(P)_{/x} \cong N(P)$. Now the lifting property is trivial for all n:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & N(P)_{/x} \\ \downarrow & \nearrow & \downarrow \cong \\ \Delta^n & \longrightarrow & N(P) \end{array}$$

Conclude by definition that $x \in N(P)$ is final. □

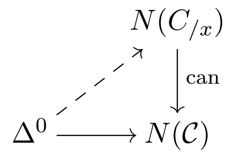
This is a special case of the slightly more general result:

Proposition 2.10. An ∞-category $N(\mathcal{C})$ of a category $\mathcal{C}$ has a final (initial) object if and only if the category has a final (initial) object.

Proof. Suppose x is a final object in the ∞-category $N(\mathcal{C})$, then x is an object in $\mathcal{C}$ and $N(\mathcal{C})_{/x} \to N(\mathcal{C})$ is a trivial fibration, that is, for every n, we have the right lifting property with respect to the inclusion $\partial\Delta^n \to \Delta^n$:

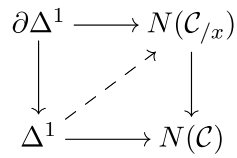
$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & N(\mathcal{C})_{/x} \\ \downarrow & \nearrow & \downarrow \text{can} \\ \Delta^n & \longrightarrow & N(\mathcal{C}) \end{array}$$

Notice that, by Proposition 1.17, $N(\mathcal{C})_{/x} \cong N(\mathcal{C}_{/x})$. Let $y \in \text{obj}(\mathcal{C})$ be any object. By the Yoneda lemma, this corresponds to a map between simplicial sets $\Delta^0 \to N(\mathcal{C})$. Consider the above commutative square with $n = 0$. Since $\partial\Delta^0 = \emptyset$, this reduces to:



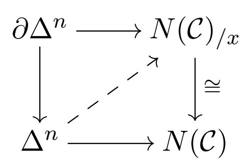
But the lifting property here tells us that $y \to x$ is an element of $C_{/x}$, that is $\hom_{\mathcal{C}}(y,x) \neq \emptyset$, and in particular, $|\hom_{\mathcal{C}}(y,x)| \geq 1$.

Let $f, g : y \to x$ be two morphisms in $\mathcal{C}$. Consider the lifting property for $n = 1$:



where $\partial\Delta^1$ takes $(0,1) \to (f,g)$ and $\Delta^1 \to N(\mathcal{C})$ represents the morphism $\mathrm{id}_y : y \to y$ in $\mathcal{C}$. This diagram clearly commutes (although, it might take a second to see what the top right corner does). Hence the lifting exists. The top triangle commuting tells us that there is a map $\alpha : y \to y$ in $\mathcal{C}_{/x}$ and the bottom triangle tells us that $\alpha = \mathrm{id}_y$. Thus, conclude that $|\hom_{\mathcal{C}}(y,x)| = 1$ and so x is a final element in $\mathcal{C}$ since y was arbitrary.

Conversely, suppose that $x \in \mathrm{obj}(\mathcal{C})$ is a final object: $|\hom_{\mathcal{C}}(y,x)| = 1$ for all $y \in \mathrm{obj}(\mathcal{C})$. We immediately have that $\mathcal{C}_{/x} \cong \mathcal{C}$ which means $N(\mathcal{C})_{/x} \cong N(\mathcal{C})$. Now the lifting property is trivial for all n:



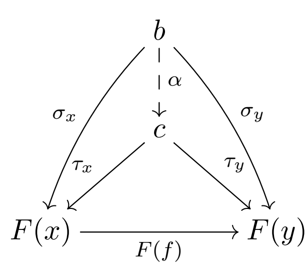
Conclude by definition that $x \in N(\mathcal{C})$ is final. $\square$

Proposition 2.11. The one-point topological space is a final object in the ∞-category of topological spaces.

Proposition 2.12. A topological space homotopy equivalent to a one point space is a final object in the ∞-category of topological spaces.

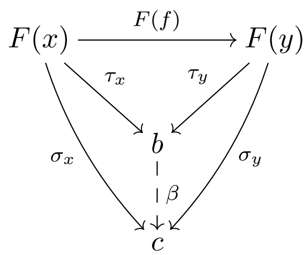
3. Limits and Colimits

Definition 3.1. Let $F : J \to \mathcal{C}$ be a functor of ordinary categories. The *limit* of F is a cone $\tau : \Delta(c) \to F$ for some $c \in \mathrm{obj}(\mathcal{C})$ such that for any other cone $\sigma : \Delta(b) \to F$ there exists a unique $\alpha : b \to c$ such that the following diagram commutes for every $f : x \to y$ in J:



We can interpret as: limits are final objects in the category of cones on F.

Definition 3.2. Let $F : J \to \mathcal{C}$ be a functor of ordinary categories. The *colimit* of F is a co-cone $\tau : F \to \Delta(b)$ for some $b \in \mathrm{obj}(\mathcal{C})$ such that for any other co-cone $\sigma : F \to \Delta(c)$. there exists a unique map $\beta : b \to c$ such that the following diagram commutes for all $f : x \to y$ in J.



We can interpret this as: colimits are initial objects in the category of co-cones on F.

Using these definitions as motiviation we are now ready to define limits and colimits for ∞-categories.

Definition 3.3. Let $\mathcal{C}$ be an ∞-category and let $p : K \to \mathcal{C}$ be an arbitrary map of simplicial sets. A *colimit* for p is an initial object of $C_{p/}$, the ∞-category of co-cones on p. A *limit* for p is a final object of $\mathcal{C}_{/p}$, the ∞-category of cones on p.

Remark 3.4. A colimit of a diagram $p : K \to \mathcal{C}$ is an object of $\mathcal{C}_{p/}$. By the Yoneda lemma, we know that

$$(\mathcal{C}_{p/})_0 \cong \hom_{\mathbf{sSet}}(\Delta^0, \mathcal{C}_{p/}) \cong \hom_p(K * \Delta^0, \mathcal{C})$$

and using our previous notation, this can be interpreted as an extension of p, $\overline{p} : K^{\triangleright} \to \mathcal{C}$.

Notation 3.5. If $p : K \to \mathcal{C}$ is a diagram, we can write $\varinjlim(p)$ to denote a colimit of p and $\varprojlim(p)$ to denote a limit of p.

Proposition 3.6. A vertex is final (initial) if and only if it is a limit (colimit) of the empty diagram.

Proof. Notice that if $p : \emptyset \to \mathcal{C}$ is the empty diagram, then

$$(\mathcal{C}_{/p})_n = \hom_p(\Delta^n * \emptyset, \mathcal{C}) = \hom_{\mathbf{sSet}}(\Delta^n, \mathcal{C}) = \mathcal{C}_n$$

which means $\mathcal{C}_{/p} = \mathcal{C}$. Now simply note that x is final in $\mathcal{C}$ if and only if x is final in $\mathcal{C}_{/p}$ if and only if x is a limit of p. □

Proposition 3.7. Limits (colimits) in 0-categories (that is, nerves of posets) are infimums (supremums).

Proof. If $r : Q \to P$ is a map of posets, then a limit of r is a final object in $N(P)_{/r} \cong N(P_{/r})$ which we know must be a final object in $P_{/r}$, but a final object in $P_{/r}$ is just the infimum of the objects involved, that is, a infimum of a sub-poset of P. □

Definition 3.8. Let $\mathcal{C}$ be an ∞-category and let $q : \square \to \mathcal{C}$ be a square. We define $\square = \Delta^1 * \Delta^1 \cong (\Lambda^2_0)^{\triangleright} \cong (\Lambda^2_2)^{\triangleleft}$.

(1) The square q is a *pushout* if $q : (\Lambda^2_0)^{\triangleright} \to \mathcal{C}$ is a colimiting cocone, that is, a colimit.
(2) The square q is a *pullback* if $q : (\Lambda^2_2)^{\triangleleft} \to \mathcal{C}$ is a limiting cone, that is, a limit.

Definition 3.9. The *homotopy pushout* of a diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \\ Z & & \end{array}$$

of topological spaces is defined as $(Y \sqcup ([0,1] \times X) \sqcup Z)$ modulo the relations $f(X) \sim \{0\} \times X$ and $g(X) \sim \{1\} \times X$.

Proposition 3.10. The homotopy pushout gives a pushout square in the ∞-category of topological spaces.

Example 3.11. Observe that the regular topological pushouts of

$$\begin{array}{ccc} S^1 & \longrightarrow & \{\text{pt}\} \\ \big\downarrow & & \\ \{\text{pt}\} & & \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 & \xrightarrow{\text{inc}} & D^2 \\ {\scriptstyle \text{inc}}\big\downarrow & & \\ D^2 & & \end{array}$$

are $\{\text{pt}\}$ and S^2 respectively. Taking the homotopy pushout results in both having the same pushout: S^2, making pushout behave well with respect to homotopy.

Definition 3.12. The *homotopy pullback* of a diagram

$$\begin{array}{ccc} & & Y \\ & & \big\downarrow {\scriptstyle g} \\ Z & \xrightarrow[f]{} & X \end{array}$$

of topological spaces is defined as

$$\{(z, \gamma, y) \in Z \times \text{top}([0,1], X) \times X : \gamma(0) = f(z),\ \gamma(1) = g(y)\}$$

Proposition 3.13. The homotopy pullback gives a pullback square in the ∞-category of topological spaces.

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