Monoidal ∞ - categories

<u>Def</u> $F: X \longrightarrow Y$ a morphism between simplicial sets is an <u>inner fibration</u>

$\Leftrightarrow$ F has the right lifting property w.r.t. $\Lambda^n_i \hookrightarrow \Delta^n$ $\forall\, 0<i<n$

i.e.: $\forall\, 0<i<n$

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "F"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

and an <u>trivial fibration</u>

$\Leftrightarrow$ F has the right lifting property w.r.t. $\partial\Delta^n \hookrightarrow \Delta^n$

i.e.:

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "F"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

<u>Def</u> <u>fibre product</u> of (simplicial) sets

For $A \xrightarrow{f} B \xleftarrow{g} C$, A, B, C sets

$$A \times_B C := \{(a,c) \in A \times C \mid f(a) = g(c)\}$$

and for $K \longrightarrow M \longleftarrow L$ K, M, L simplicial sets

$(K \times_M L)_q = K_n \times_{M_n} L_n$ defines a simplicial set.

<u>Def</u> Let $p: X \longrightarrow S$ be an inner fibration between ∞ - categories / simplicial sets

and $f: \Delta^1 \longrightarrow X$ an edge with source $x: \Delta^0 \longrightarrow X$ i.e. $\begin{tikzcd} \Delta^0 \arrow[r, hook] \arrow[rr, bend left, "x"] & \Delta^1 \arrow[r, "f"] & X \end{tikzcd}$

f is then called <u>p-coCartesian</u>

$\Leftrightarrow$ $X_{f/} \longrightarrow X_x \times_{S_{p(x)/}} S_{p(f)/}$ is a trivial fibration

unwraping this definition

$\Leftrightarrow$ $\forall n \geq 2$

$$\begin{tikzcd}
\Delta^1 \arrow[d, hook] \arrow[dr, "f"] & \\
\Lambda^n_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

Def Let $p: X \longrightarrow S$ be a map of simplicial sets

p is called a coCartesian fibration

$:\Longleftrightarrow$ (i) p is an inner fibration

(ii) For every vertex $\tilde{x}$ of X
and every edge $f: x \longrightarrow y$ of S with $p(\tilde{x}) = x$
there is a p-coCartesian edge $\tilde{f}: \tilde{x} \longrightarrow \tilde{y}$ with $p(\tilde{f}) = f$

Remark

The ∞-category of coCartesian fibrations towards an ∞-category S is equivalent to the ∞-category of functors from S to the ∞-category of ∞-categories.

Def For $p: C \longrightarrow D$ a morphism between ∞-categories
we can interpret an object $d \in D_0$ as simplicial set by using degeneracy morphisms.
Then we define the fiber $C_d := p^{-1}(d)$

Remark

p is an inne fibration $\Longleftrightarrow$ the fibe of p ove any simplex is an ∞-category.

Def (The ∞-category $N\Delta^{op}$)

A q-simplex of $N\Delta^{op}$ is a sequence of non-decreasing morphisms $[m_0] \leftarrow \dots \leftarrow [m_q]$.
Face maps remove an $[m_i]$ and compose the morphisms and degeneracy maps insert an identity morphism.

Def A monoidal ∞-category is a coCartesian fibration $p: M^{\otimes} \longrightarrow N\Delta^{op}$
s.t. for each $n \geq 0$ the inclusions $\phi^i : [1] \longrightarrow [n]$, $\{0,1\} \longmapsto \{i-1, i\}$ induce an equivalence

$$M^{\otimes}_{[n]} \to M^{\otimes}_{\{0,1\}} \times \dots \times M^{\otimes}_{\{n-1,n\}} \xrightarrow{\sim} \left(M^{\otimes}_{[1]}\right)^n$$

$M := M^{\otimes}_{[1]}$ is called the underlying ∞-category

Exa. 1

Let G be a group
Define the simplicial set $G^{\otimes}$ as follows:
a q-simplex of $G^{\otimes}$ is a tuple

$$\left([m_0] \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_q} [m_q], \left((g_{0,1}, \dots, g_{0,m_0}), \dots, (g_{q,1}, \dots, g_{q,m_q}) \right) \right)$$

s.t.: $\forall\, 0 < i \leq q$ and $0 < j \leq m_i$

$$g_{i,j} = g_{i-1,\alpha_i(j)}\, g_{i-1,\alpha_i(j)-1} \cdots g_{i-1,\alpha_i(j-1)+1}$$

if $\alpha_i(j-1) = \alpha_i(j)$
then $g_{i,j} = e \in G$ the identity

Define $p: G^{\otimes} \longrightarrow N\Delta^{op}$ as the projection to the first element

Observation: $p^{-1}([1]) = G^{\otimes}_{[1]} = \{ ([1], ((g_{0,1}))) \mid g_{0,1} \in G \} = G$

because $q = 0$, $m_0 = m_q = 1$

Observation: The preimage of $\delta_1 : [1] \longrightarrow [2]$, $0 \longmapsto 0$, $1 \longmapsto 2$ gives us the multiplication:

$$p^{-1}(\delta_1) = \left\{ \left([2] \xleftarrow{\delta_1} [1], ((g_{0,1}, g_{0,2}), (g_{1,1})) \right) \;\middle|\; g_{0,1}, g_{0,2} \in G \,.\, g_{1,1} = g_{0,1} g_{0,2} \right\}$$

And $(G^{\otimes}_{[1]})^n = \{ ([1], ((g_{0,1}))) \mid g_{0,1} \in G \}^n \overset{\phi^i}{\cong} \{ (\{0,1\}, ((g_{0,1}))) \mid g_{0,1} \in G \} \times \dots \times \{ (\{i-1, i\}, ((g_{0,1}))) \mid g_{0,1} \in G \}$

$\cong$ Isomorphic

$\{ ([n], ((g_{0,1}, \dots, g_{0,n}))) \mid g_{0,i} \in G\ \forall i \} = G^{\otimes}_{[n]}$

Exa 2.

Let (X,x) be a pointed topological space

Define an ∞-category $\Omega X^{\otimes}$ as follows:

A q-simplex is a tuple

$$\left([m_0] \xleftarrow{\alpha_1} \dots \xleftarrow{\alpha_q} [m_q],\ \left(h_I : \Delta^j_{top} \times \Delta^{m_{i_j}}_{top} \longrightarrow X\right)_{I=(i_0 \le \dots \le i_j) \subseteq [q]}\right)$$

s.t.: 1. for each corne $e_k = (0,..,0,\underset{\substack{\uparrow \\ k}}{1},0,..,0) \in \Delta^{m_{i_j}}_{top}$

$$h_I(\Delta^j_{top} \times \{e_k\}) = x$$

2. For every inclusion $I' = (i_0' \le \dots \le i'_{j'}) \subseteq I \subseteq [q]$

$$h_I \circ (\Delta^{j'}_{top} \times \Delta^{m_{i'_{j'}}}_{top} \longrightarrow \Delta^j_{top} \times \Delta^{m_{i_j}}_{top}) = h_{I'}.$$

where $\Delta^{j'}_{top} \longrightarrow \Delta^j_{top}$ is induced by $I' \subseteq I$

and $\Delta^{m_{i'_{j'}}}_{top} \longrightarrow \Delta^{m_{i_j}}_{top}$ is induced by $\alpha_{i'_{j'}} \alpha_{i'_{j'}+1} \dots \alpha_{i_j} : [m_{i_j}] \longrightarrow [m_{i'_{j'}}]$

$\Omega X^{\otimes} \longrightarrow N\Delta^{op}$ is again the projection to the first component.

Observation

A q-simplex of $\Omega X^{\otimes}_{[1]}$ is a tuple

$$\left(\underset{\substack{\uparrow \\ m_0}}{[1]} \xleftarrow{id} \dots \xleftarrow{id} \underset{\substack{\uparrow \\ m_q}}{[1]},\ h_I : \Delta^j_{top} \times \Delta^1_{top} \longrightarrow X\right)_{I=(i_0 \le \dots \le i_j) \subseteq [q]}$$

s.t.: $h_I(\Delta^j_{top} \times \{(1,0)\}) = x$ | so for each $y \in \Delta^j_{top}$ we get $\overset{\text{a set of}}{\text{a}}$ loops $h_I(y,t) : \Delta^1_{top} \to X$

$h_I(\Delta^j_{top} \times \{(0,1)\}) = x$ | therefore $\Delta^j_{top} \longrightarrow \Omega X$ which is just $(\mathrm{Sing}\,\Omega X)_q$

and for $I' \subseteq I$ $\quad h_I \circ (\Delta^{j'}_{top} \times \Delta^1_{top} \longrightarrow \Delta^j_{top} \times \Delta^1_{top}) = h_{I'}$

Similarly we can obtain that the fibre over $[n]$ is the singuler simplicial set of a subspace of $\hom(\Delta^n_{top}, X)$ where corners get send to x.

This time we get a homotopy equivalence $\Omega X^{\otimes}_{[n]} \xrightarrow{\sim} \left(\Omega X^{\otimes}_{[1]}\right)^n$

Def Let $C^{\otimes} \xrightarrow{p} N\Delta^{op}$ and $D^{\otimes} \xrightarrow{q} N\Delta^{op}$ be monoidal ∞-categories

A functor $F : C^{\otimes} \longrightarrow D^{\otimes}$ is <u>monoidal</u>

$:\Leftrightarrow$ The diagram
$$\begin{tikzcd} C^{\otimes} \arrow[rr, "F"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & D^{\otimes} \arrow[dl, "q"] \\ & N\Delta^{op} & \end{tikzcd}$$
commutes

and p-coCartesian morphisms become q-coCartesian via F.

Def: A morphism $f : [m] \to [n]$ is <u>convex</u> if f is injective

and the image $\{f(0), \dots, f(m)\} \subseteq [n]$ is a convex subset.

F is <u>lax-monoidal</u> $\Leftrightarrow$
$$\begin{tikzcd} C^{\otimes} \arrow[rr, "F"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & D^{\otimes} \arrow[dl, "q"] \\ & N\Delta^{op} & \end{tikzcd}$$

and for every p-coCartesian morphism α in $C^{\otimes}$

s.t.: $p(\alpha)$ is a convex morphism

$F(\alpha)$ is q-coCartesian

Def For a monoidal ∞-category C.

An <u>algebra object</u> of C is a lax monoidal functor $N\Delta^{op} \longrightarrow C^{\otimes}$