**Exercise.** Let $p : X \to S$ be a morphism of simplicial sets, and $f : \Delta^1 \to X$ an edge, with source $x : \Delta^0 \to X$ (that is, $x$ is the composition of $f$ with the canonical inclusion $\Delta^0 \subseteq \Delta^1$). Show that the following are equivalent.

(A) The induced map $X_{/f} \to X_x \times_{S_{/p(x)}} S_{/p(f)}$ is a trivial Kan fibration.

(B) For every $n \geq 2$, and every commutative diagram

$$\begin{tikzcd}
\Delta^1 \arrow[d] \arrow[dr, "f"] & \\
\Lambda^n_0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

there is a dashed morphism making the diagram commutative.

*Step a).* Recall that a morphism is a trivial Kan fibration if and only if it has the left lifting property with respect to the morphisms $\partial\Delta^n \to \Delta^n$ for all $n \geq 0$. So (A) is equivalent to:

(A') For all $n \geq 0$ and every commutative diagram

(1)

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & X_{/f} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & X_x \times_{S_{/p(x)}} S_{/p(f)}
\end{tikzcd}$$

there is a dashed morphism making the diagram commutative.

*Step b).* Recall that by the universal property of overcategories, the upper horizontal morphism in (1) corresponds to a unique commutative diagram

(2)

$$\begin{tikzcd}
\Delta^1 \arrow[r] \arrow[rr, bend left, "f"] & \Delta^1 * \partial\Delta^n \arrow[r] & X
\end{tikzcd}$$

where $\Delta^1 \subseteq \Delta^1 * \partial\Delta^n$ is the canonical inclusion (and conversely, such a diagram corresponds to a unique morphism $\partial\Delta^n \to X_{/f}$). Moreover, by the universal property of fibre products, the lower horizontal morphism in (1) corresponds to a unique commutative diagram

$$\begin{tikzcd}
\Delta^n \arrow[r] \arrow[d] & S_{/p(f)} \arrow[d] \\
X_x \arrow[r] & S_{/p(x)}
\end{tikzcd}$$

(where the lower morphism and right morphism are the canonical ones), and by the universal property of overcategories, such a diagram corresponds uniquely to a commutative diagram

(3)
$$\begin{tikzcd}
\Delta^0 \arrow[r] \arrow[rr, bend left, "x"] & \Delta^0 * \Delta^n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^1 \arrow[r] \arrow[rr, bend right, "p(f)"'] & \Delta^1 * \Delta^n \arrow[r] & S
\end{tikzcd}$$

where $\Delta^0 \subseteq \Delta^0 * \partial\Delta^n$, $\Delta^1 \subseteq \Delta^1 * \partial\Delta^n$, $\Delta^0 * \Delta^n \subseteq \Delta^1 * \Delta^n$ are the canonical ones.

*Step c).* Observe that the commutivity of (1) (without the dashed morphism) is equivalent to the requirement that (2) and (3) fit together into a commutative diagram

(4)
$$\begin{tikzcd}
\Delta^1 \arrow[d] \arrow[dr, "f"] & \\
(\Delta^1 * \partial\Delta^n) \cup (\Delta^0 * \Delta^n) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^1 * \Delta^n \arrow[r] & S
\end{tikzcd}$$

*Step d).* Observe that the inclusion

$$(\Delta^1 * \partial\Delta^n) \cup (\Delta^0 * \Delta^n) \subseteq \Delta^1 * \Delta^n$$

is canonically isomorphic to the inclusion

$$\Lambda_0^{n+2} \subseteq \Delta^{n+2}.$$

(Indeed, $(\Delta^0 * \Delta^n) \subseteq \Delta^1 * \Delta^n$ is the inclusion of the face $d_1\Delta^{n+2} \subseteq \Delta^{n+2}$. Moreover, $\partial\Delta^n$ is the union of the faces $d_i\Delta^n \subseteq \Delta^n$ for $0 \le i \le n$, and $\Delta^1 * d_i\Delta^n \subseteq \Delta^{n+2}$ is the inclusion of the face $d_{i+2}\Delta^{n+2} \subseteq \Delta^{n+2}$). Conclude that (4) is isomorphic to

(5)
$$\begin{tikzcd}
\Delta^1 \arrow[d] \arrow[dr, "f"] & \\
\Lambda_0^{n+2} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^{n+2} \arrow[r] & S
\end{tikzcd}$$

*Step e).* Conclude that (1) admits a dashed morphism making the diagram commute if and only if the corresponding diagram

(6)
$$\begin{tikzcd}
\Delta^1 \arrow[d] \arrow[dr, "f"] & \\
\Lambda_0^{n+2} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^{n+2} \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a dashed morphism making the diagram commute.